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WEAKLY COMMUTING MAPPINGS AND ITS VARIANTS FOR GENERALIZED ψ −WEAK CONTRACTION IN METRIC SPACES

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Abstract. In this paper, we prove common fixed point theorems for weakly commuting and its variants point wise R-weakly commuting and reciprocal continuous mappings, R-weakly commuting mappings of type (P) using generalized ψ −weak contraction condition that involves cubic and quadratic terms of distance function $d(x, y)$.

Keywords and phrases: ψ −weak contraction; weakly commuting mappings; R-weakly commuting; R-weakly commuting of type (P); point wise R-weakly commuting; reciprocal continuous mappings.

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1. INTRODUCTION

The Banach Contraction principle has been generalized in various ways such as by relaxing continuity, extending the number of mappings, using control functions and soothing minimal commutative conditions and various properties. Banach's contraction principle is the primary tool for finding fixed points of all contraction type maps. It has a constructive proof which makes the theorem worthy as it yields an algorithm for fixed computing points.

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It was a turning point in fixed point theory literature when the notion of commutative mappings was introduced. On the other hand, Sessa [16] coined the notion of weak commutativity of mappings. In 1986, a major breakthrough was done when Jungck [5] introduced the notion of compatibility. This notion is useful to find common fixed point for pairs of mappings assuming continuity of at least one of the mappings.

The study of common fixed point theorem from the class of compatible mappings to non compatible mappings was initiated by Pant [12,13] with the introduction of notion of R-weakly commuting. In this paper, first we introduce generalized $\psi-$weak contraction condition for a pair of map that involves cubic and quadratic terms of distance function $d(x, y)$ and proved a common fixed point theorem for weakly commuting mappings and its variants.

2. PRELIMINARIES

Banach fixed point theorem states that every contraction mapping on a complete metric space has a unique fixed point. Let (X, d) be a complete metric space. If $T : X \to X$ satisfies $d(T(x), T(y)) \leq k(d(x, y))$ for all $x, y \in X, 0 \leq k < 1,$ then it has a unique fixed point.

In 1969, Boyd and Wong [3] replaced the constant k in Banach contraction principle by a control function ψ as follows:

Let (X, d) be a complete metric space and $\psi : [0, \infty) \to [0, \infty)$ be upper semi continuous from the right such that $0 \leq \psi(t) < t$ for all $t > 0$.

If $T : X \to X$ satisfies $d(T(x), T(y)) \leq \psi(d(x, y))$ for all $x, y \in X$, then it has a unique fixed point.

Definition 2.1 Two self mappings f and g of a metric space (X, d) are said to be commuting if $fgx = gfx$ for all x in X.

The first ever attempt to relax the commutativity of mappings to weak commutative was initiated by Sessa [16] in 1982 as follows:

Definition 2.2 [16] Two self mappings f and g of a metric space (X, d) are said to be weakly commuting if $d(fgx, gfx) \leq d(gx, fx)$ for all x in X.

Remark 2.1 [16] Commutative mappings are weak commutative mappings, but converse may not be true.

In 1994, Pant [12] introduced the notion of R-weakly commuting mappings in metric spaces to widen the scope of finding fixed point for mappings from class of compatible maps to a wider class of R-weakly commuting mappings. These maps are not necessarily continuous at fixed point.

Definition 2.3[12] A pair (f, g) of self-mappings of a metric space (X, d) is said to be R-weakly commuting if there exists some real number $R > 0$ such that

$$d(fgx, gfx) \leq Rd(fx, gx), \text{ for all } x \in X.\text{''}$$

Remark 2.2[12] (i) “For $R = 1$, every R -weakly commuting pair is weakly commuting.

(ii) Weak commutativity implies R-weak commutativity. However, R-weak commutativity implies weak commutativity only when $R \leq 1$.”

In 1998, Pant [13] investigated the existence of common fixed points for non-compatible mappings and point wise R- weak Commutativity.

Definition 2.5[13] A pair (f, g) of self-mappings of a metric space (X, d) is said to be point wise R-weakly commuting on X, iff given $x \in X$, there exists $R > 0$ such that

$$d(fgx, gfx) \leq Rd(fx, gx).$$

Remark 2.3 It is clear from the definition 2.5 that f and g can fail to be pointwise R-weakly commuting only if there exists some x in X such that $fx = gx$ but $fgx \neq gfx$, i.e., only if they possess a coincidence point at which they do not commute.

In 1999, Pant [14] introduced a new type of continuity condition, known as reciprocal continuity defined as follows:

Definition 2.6 [14] A pair (f, g) of self-mappings of a metric space (X, d) is said to be reciprocally continuous if $\lim_{n\to\infty} fgx_n = fz$ and $\lim_{n\to\infty} gfx_n = gz$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} d(fx_n, z) = \lim_{n\to\infty} d(gx_n, z) = 0$ for some $z \in X$.

In 1982, S. Sessa [16] generalized the concept of commutativity to the notion of weak commutativity of maps. Thereafter, in 1986, Jungck [5] generalized and extend the notion of weak commutativity to compatible mappings.

Definition 2.7[5] Two self mappings f and g of a metric space (X, d) are called compatible if

$$lim_n d(fgx_n, gfx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$lim_n fx_n = lim_n gx_n = t, \text{ for some } t \text{ in } X.$$

Now we state some properties for compatible mappings that are fruitful for further study.

Proposition 2.1[5] Let S and T be compatible mappings of a metric space (X, d) into itself. If $St = Tt$ for some $t \in X$, then $STt = SSt = TTt = TSt$.

Proposition 2.2 [5] Let S and T be compatible mappings of a metric space (X, d) into itself. Suppose that $lim_n Sx_n = lim_n Tx_n = t$ for some t in X. Then the following holds:

(i) $lim_n TSx_n = St$ if S is continuous at t;

(ii) $lim_n STx_n = Tt$ if T is continuous at t;

(iii) $STt = TSt$ and $St = Tt$ if S and T are continuous at t.

Definition 2.8[17] A pair of self-mappings (f, g) of a metric space (X, d) is said to be R-weakly commuting mappings of type (P) if there exists some $R > 0$ such that

$$d(ffx, ggx) \leq Rd(fx, gx) \text{ for all } x \in X.$$

Remark 2.4[17] R- weakly commuting mappings, R-weakly commuting of type (A_f), R-weakly commuting of type (A_g) and R-weakly commuting of type (P) are distinct.

3. MAIN RESULTS

In 2013, Murthy and Prasad [10] introduced a new type of inequality for a map that involves cubic terms of metric function $d(x, y)$ that extended and generalized the results of Alber and Gueree-Delabriere [1] and many others cited in the literature of fixed point theory.

Now we introduce the generalized ψ - weak contraction of Murthy and Prasad [10] for a pairs of mappings in the following way:

Let A, B, S and T are self mappings on a metric space (X, d) satisfying the following conditions:

(C1) $S(X) \subset B(X), T(X) \subset A(X)$;

(C2) $d^3(Sx, Ty) \leq \psi \begin{Bmatrix} \mathrm{d}^2(\mathrm{Ax}, \mathrm{Sx})\mathrm{d}(\mathrm{By}, \mathrm{Ty}), \\ \mathrm{d}(\mathrm{Ax}, \mathrm{Sx})\mathrm{d}^2(\mathrm{By}, \mathrm{Ty}), \\ \mathrm{d}(\mathrm{Ax}, \mathrm{Sx})\mathrm{d}(\mathrm{Ax}, \mathrm{Ty})\mathrm{d}(\mathrm{By}, \mathrm{Sx}), \\ \mathrm{d}(\mathrm{Ax}, \mathrm{Ty})\mathrm{d}(\mathrm{By}, \mathrm{Sx})\mathrm{d}(\mathrm{By}, \mathrm{Ty}) \end{Bmatrix}$

for all $x, y \in X$, where ψ: $[0, \infty) \rightarrow [0, \infty)$ is a continuous and non-decreasing function with $\psi(t) < t$ for each $t > 0$.

Now we prove fixed point theorems for a pair of weakly commuting mappings and its variants point wise R- weakly commuting and reciprocal continuous mappings, R- weakly commuting mappings of type (P) involves cubic terms and product of quadratic and linear terms of distance function $d(x, y)$.

Theorem 3.1 Let (X, d) be a complete metric space. Let S, T, A and B be four self mappings of a complete metric space (X, d) satisfying (C1), (C2) and the following conditions:

(3.1) one of S, T, A and B is continuous;

Further, assume that the pairs (A, S) and (B, T) are weakly commuting. Then S, T, A and B have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be an arbitrary point. From (C1) we can find x_1 such that $S(x_0) = B(x_1) = y_0$ for this x_1 one can find $x_2 \in X$ such that $T(x_1) = A(x_2) = y_1$.Continuing in this way, one can construct a sequence $\{y_n\}$ in X such that

$y_{2n} = S(x_{2n}) = B(x_{2n+1})$,

$y_{2n+1} = T(x_{2n+1}) = A(x_{2n+2})$ for each $n \geq 0$.

For brevity, one can denote $\alpha_{2n} = d(y_{2n}, y_{2n+1})$.

First we prove that $\{\alpha_{2n}\}$ is non increasing sequence and converges to zero.

Case I. If n is even, taking $x = x_{2n}$ and $y = x_{2n+1}$ in (C2), we get

$$d^3(Sx_{2n}, Tx_{2n+1}) \leq \psi \begin{Bmatrix} d^2(Ax_{2n}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \\ d(Ax_{2n}, Sx_{2n})d^2(Bx_{2n+1}, Tx_{2n+1})' \\ d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tx_{2n+1})d(Bx_{2n+1}, Sx_{2n}), \\ d(Ax_{2n}, Tx_{2n+1})d(Bx_{2n+1}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \end{Bmatrix}$$

Using (3.1), we have

$$d^3(y_{2n}, y_{2n+1}) \leq \psi \begin{Bmatrix} d^2(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1}) \\ d(y_{2n-1}, y_{2n})d^2(y_{2n}, y_{2n+1})' \\ d(y_{2n-1}, y_{2n})d(y_{2n-1}, y_{2n+1})d(y_{2n}, y_{2n}), \\ d(y_{2n-1}, y_{2n+1})d(y_{2n}, y_{2n})d(y_{2n}, y_{2n+1}) \end{Bmatrix} \tag{3.2}$$

On using $\alpha_{2n} = d(y_{2n}, y_{2n+1})$ in (3.2), we have

$$\alpha_{2n}^3 \leq \psi \{\alpha_{2n-1}^2\alpha_{2n}, \alpha_{2n-1}\alpha_{2n}^2, 0,0\} \tag{3.3}$$

By triangular inequality, we get

$$d(y_{2n-1}, y_{2n+1}) \leq d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})$$

$$= \alpha_{2n-1} + \alpha_{2n}$$

If $\alpha_{2n-1} < \alpha_{2n}$ and using property of ψ, then (3.3) reduces to

$\alpha_{2n}^3 < \alpha_{2n}^3$, a contradiction, therefore, $\alpha_{2n} \leq \alpha_{2n-1}$.

Case II. If n is odd, then in a similar way, one can obtain $\alpha_{2n+1} \leq \alpha_{2n}$.

It follows that the sequence $\{\alpha_{2n}\}$ is decreasing.

Let $\lim_{n\to\infty} \alpha_{2n} = r$, for some $r \geq 0$.

Suppose $r > 0$, then from inequality (C2), we have

$$d^3(Sx_{2n}, Tx_{2n+1}) \leq \psi \begin{Bmatrix} \{d^2(Ax_{2n}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1}), \\ d(Ax_{2n}, Sx_{2n})d^2(Bx_{2n+1}, Tx_{2n+1}) \\ . \\ d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tx_{2n+1})d(Bx_{2n+1}, Sx_{2n}), \\ d(Ax_{2n}, Tx_{2n+1})d(Bx_{2n+1}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \end{Bmatrix}$$

Now by using (3.3), triangular inequality, property of ψ and proceed limits $n \to \infty$, we get

$r^3 \leq \psi\,(r^3) < r^3$, a contradiction, therefore we get $r = 0$. Therfore

$$\lim_{n\to\infty} \alpha_{2n} = \lim_{n\to\infty} d(y_{2n}, y_{2n+1}) = r = 0. \tag{3.4}$$

Now we show that $\{y_n\}$ is a Cauchy sequence. Suppose we assume that $\{y_n\}$ is not a Cauchy sequence. For given $\epsilon > 0$ we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k , $n(k) > m(k) > k$.

$$d(y_{m(k)}, y_{n(k)}) \geq \epsilon, d\big(y_{m(k)}, y_{n(k)-1}\big) < \epsilon \tag{3.5}$$

Now $\quad \epsilon \leq d\big(y_{m(k)}, y_{n(k)}\big) \leq d\big(y_{m(k)}, y_{n(k)-1}\big) + d(y_{n(k)-1}, y_{n(k)})$.

Letting $k \to \infty$, we get $\lim_{k\to\infty} d\big(y_{m(k)}, y_{n(k)}\big) = \epsilon$

Now from the triangular inequality we have,

$$\big|d\big(y_{n(k)}, y_{m(k)+1}\big) - d\big(y_{m(k)}, y_{n(k)}\big)\big| \le d\big(y_{m(k)}, y_{m(k)+1}\big).$$

Taking limits as $k \to \infty$ and using (3.4) and (3.5), we have

$\lim_{k\to\infty} d\big(y_{n(k)}, y_{m(k)+1}\big) = \epsilon.$

Again from the triangular inequality, we have

$$\big|d\big(y_{m(k)}, y_{n(k)+1}\big) - d\big(y_{m(k)}, y_{n(k)}\big)\big| \le d\big(y_{n(k)}, y_{n(k)+1}\big).$$

Taking limits as $k \to \infty$ and using (3.4) and (3.5), we have

$\lim_{k\to\infty} d\big(y_{m(k)}, y_{n(k)+1}\big) = \epsilon.$

Similarly on using triangular inequality, we have

$$\big|d\big(y_{m(k)+1}, y_{n(k)+1}\big) - d\big(y_{m(k)}, y_{n(k)}\big)\big| \le d\big(y_{m(k)}, y_{m(k)+1}\big) + d\big(y_{n(k)}, y_{n(k)+1}\big).$$

Taking limit as $k \to \infty$ in the above inequality and using (3.4) and (3.5), we have

$$\lim_{k\to\infty} d\big(y_{n(k)+1}, y_{m(k)+1}\big) = \epsilon.$$

On putting $x = x_{m(k)}$ and $y = x_{n(k)}$ in (C2), we get

$$d^3\big(Sx_{m(k)}, Tx_{n(k)}\big) \le \psi \left\{ \begin{array}{c} d^2\big(Ax_{m(k)}, Sx_{m(k)}\big) d\big(Bx_{n(k)}, Tx_{n(k)}\big), \\ d\big(Ax_{m(k)}, Sx_{m(k)}\big) d^2\big(Bx_{n(k)}, Tx_{n(k)}\big) \\ d\big(Ax_{m(k)}, Sx_{m(k)}\big) d\big(Ax_{m(k)}, Tx_{n(k)}\big) d\big(Bx_{n(k)}, Sx_{m(k)}\big), \\ d\big(Ax_{m(k)}, Tx_{n(k)}\big) d\big(Bx_{n(k)}, Sx_{m(k)}\big) d\big(Bx_{n(k)}, Tx_{n(k)}\big) \end{array} \right\}$$

Using (3.1), we obtain

$$d^3\big(y_{m(k)}, y_{n(k)}\big) \le \psi \left\{ \begin{array}{c} d^2\big(y_{m(k)-1}, y_{m(k)}\big) d\big(y_{n(k)-1}, y_{n(k)}\big), \\ d\big(y_{m(k)-1}, y_{m(k)}\big) d^2\big(y_{n(k)-1}, y_{n(k)}\big) \\ d\big(y_{m(k)-1}, y_{m(k)}\big) d\big(y_{m(k)-1}, y_{n(k)}\big) d\big(y_{n(k)-1}, y_{m(k)}\big), \\ d\big(y_{m(k)-1}, y_{n(k)}\big) d\big(y_{n(k)-1}, y_{m(k)}\big) d\big(y_{n(k)-1}, y_{n(k)}\big) \end{array} \right\}$$

Letting $k \to \infty$, and using property of ψ and $\emptyset$, we have

$\epsilon^3 \le 0$, which is a contradiction.

Thus $\{y_n\}$ is a Cauchy sequence in X.

Since (X, d) is a complete metric space, therefore, $\{y_n\}$ converges to a point z as $n \to \infty$. Consequently, the subsequences $\{Sx_{2n}\}, \{Ax_{2n}\}, \{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ also converges to the same point z.

Case 1. Suppose that A is continuous. Then $\{AAx_{2n}\}$and $\{ASx_{2n}\}$ converges to Az as $n \to \infty$. Since the mappings, A and S are weakly commuting on X, therefore,

$$d(ASx_{2n}, SAx_{2n}) \leq d(Sx_{2n}, Ax_{2n}),$$

Proceeding limit as $n \to \infty$,

we get $\lim_{n\to\infty} d(SAx_{2n}, Az) \leq d(z, z) = 0$, i.e., $\lim_{n\to\infty} SAx_{2n} = Az$.

Now we show that $z = Az$. On putting $x = Ax_{2n}$ and $y = x_{2n+1}$ in (C2) we get

$$d^3(SAx_{2n}, Tx_{2n+1}) \leq \psi \left\{ \begin{matrix} d^2(AAx_{2n}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \\ d(AAx_{2n}, SAx_{2n})d^2(Bx_{2n+1}, Tx_{2n+1}), \\ d(AAx_{2n}, SAx_{2n})d(AAx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SAx_{2n}), \\ d(AAx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \end{matrix} \right\}$$

Proceeding limit as $n \to \infty$, we get

$d^3(Az, z) \quad \leq \psi\{0,0,0,0\}$,

Or $d^3(Az, z) \leq 0$

On simplification, and using the properties of ψ, we get $d^2(Az, z) = 0$.

This implies that $Az = z$.

Next, we will show that $Sz = z$.

On putting $x = z$ and $y = x_{2n+1}$ in (C2) we have

$$d^3(Sz, Tx_{2n+1}) \leq \psi \left\{ \begin{matrix} d^2(Az, Sz)d(z, z) \\ , d(Az, Sz)d^2(z, z), \\ d(Az, Sz)d(Az, z)d(z, Sz), \\ d(Az, z)d(z, Sz)d(z, z) \end{matrix} \right\}$$

Passing limit as $n \to \infty$, and after simplification, using the properties of ψ, we have

$d^3(Sz, z) \leq \psi\{0,0,0,0\} = 0$.

Thus we get $d^3(Sz, z) = 0$. This implies that $Sz = z$. Since $S(X) \subset B(X)$, therefore, there exists a point $u \in X$ such that $z = Sz = Bu$.

Now we show that $z = Tu$.

For this we put $x = z$ and $y = u$ in (C2) we get

$$d^3(Sz, Tu) \leq \psi \begin{Bmatrix} d^2(Az, Sz)d(Bu, Tu), \\ d(Az, Sz)d^2(Bu, Tu) \\ d(Az, Sz)d(Az, Tu)d(Bu, Sz), \\ d(Az, Tu)d(Bu, Sz)d(Bu, Tu) \end{Bmatrix}$$

On solving, we get

$$d^3(z, Tu) \leq \ \psi \begin{Bmatrix} d^2(z, z)d(z, Tu), \\ d(z, z)d^2(z, Tu), \\ d(z, z)d(z, Tu)d(z, z), \\ d(z, Tu)d(z, z)d(z, Tu) \end{Bmatrix}$$

This implies that $z = Tu$. Since the pair (B, T) is weak commutative, therefore, we have

$$d(Bz, Tz) = d(BTu, TBu) \leq \ d(Bu, Tu) = d(z, z) = 0.$$

Thus $Bz = Tz$.

From (C2) we have

$$d^3(Sz, Tz) \leq \ \psi \begin{Bmatrix} d^2(Az, Sz)d(Bz, Tz), \\ d(Az, Sz)d^2(Bz, Tz) \\ d(Az, Sz)d(Az, Tz)d(Bz, Sz), \\ d(Az, Tz)d(Bz, Sz)d(Bz, Tz) \end{Bmatrix}$$

Therefore, we get

$d^3(z, Tz) \leq \ \psi\{0,0,0,0\}$

Using the properties of ψ , we have $z = Tz$.

Case 2. Suppose that B is continuous; we can obtain the same result by way of Case 1.

Case 3. Suppose that S is continuous.

Then $\{SSx_{2n}\}$and $\{SAx_{2n}\}$ converges to Sz as $n \to \infty$. Since the mappings A and S are weakly commuting on X, therefore, we have

$$d(ASx_{2n}, SAx_{2n}) \ \leq \ d(Sx_{2n}, Ax_{2n})$$

Proceeding limit as $n \to \infty$, we get $\lim_{n\to\infty} d(ASx_{2n}, Sz) \leq d(z, z) = 0$,

i.e., $\lim_{n\to\infty} ASx_{2n} = Sz$.

Now we prove that $z = Sz$.

For this put $x = Sx_{2n}$ and $y = x_{2n+1}$ in (C2), we get

$$d^3(SSx_{2n}, Tx_{2n+1}) \leq \psi \left\{ \begin{matrix} d^2(ASx_{2n}, SSx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \\ , d(ASx_{2n}, SSx_{2n})d^2(Bx_{2n+1}, Tx_{2n+1}) \\ d(ASx_{2n}, SSx_{2n})d(ASx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SSx_{2n}), \\ d(ASx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SSx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \end{matrix} \right\}$$

Taking limit as $n \to \infty$, and using the properties of ψ we have

$d^3(Sz, z) \leq \psi\{0,0,0,0\}$

or $d^3(Sz, z) \leq 0$

Thus we get $d^2(Sz, z) = 0$. This implies that $Sz = z$.

Since $S(X) \subset B(X)$ and hence there exists a point $v \in X$ such that $z = Sz = Bv$.

Now we claim that $z = Tv$.

For this we put $x = Sx_{2n}$ and $y = v$ in (C2) we get

$$d^3(SSx_{2n}, Tv) \leq \psi \left\{ \begin{matrix} d^2(ASx_{2n}, SSx_{2n})d(Bv, Tv) \\ , d(ASx_{2n}, SSx_{2n})d^2(Bv, Tv), \\ d(ASx_{2n}, SSx_{2n})d(ASx_{2n}, Tv)d(Bv, SSx_{2n}), \\ d(ASx_{2n}, Tv)d(Bv, SSx_{2n})d(Bv, Tv) \end{matrix} \right\}$$

Taking limit as $n \to \infty$, and

$$d^3(z, Tv) \leq \psi \left\{ \begin{matrix} d^2(z, z)d(z, Tv), \\ d(z, z)d^2(z, Tv), \\ d(z, z)d(z, Tv)d(z, z), \\ d(z, Tv)d(z, z)d(z, Tv) \end{matrix} \right\}$$

Using the properties of ψ , we have $z = Tv$. Since the pair (B, T) is weakly commuting on X, so we have $d(TBv, BTv) \leq d(Tv, Bv) = d(z, z) = 0$.

So, $Bz = Tz$.

Now put $x = x_{2n}$ and $y = z$ in (C2), we get

$$d^3(Sx_{2n}, Tz) \leq \psi \left\{ \begin{matrix} d^2(Ax_{2n}, Sx_{2n})d(Bz, Tz) \\ , d(Ax_{2n}, Sx_{2n})d^2(Bz, Tz), \\ d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tz)d(Bz, Sx_{2n}), \\ d(Ax_{2n}, Tz)d(Bz, Sx_{2n})d(Bz, Tz) \end{matrix} \right\}$$

Passing limit as $n \to \infty$, we get

$d^3(z, Tz) \leq \psi\{0,0,0,0\}$ using the properties of ψ , we have $z = Tz$.

Since $T(X) \subset A(X)$, therefore, there exists a point $w \in X$ such that $z = Tz = Aw$.

We claim that $z = Sw$.

For this, we put $x = w$ and $y = z$ in (C2) we have

$$d^3(Sw,Tz) \leq \psi \begin{Bmatrix} d^2(Aw,Sw)d(Bz,Tz), \\ d(Aw,Sw)d^2(Bz,Tz), \\ d(Aw,Sw)d(Aw,Tz)d(Bz,Sw), \\ d(Aw,Tz)d(Bz,Sw)d(Bz,Tz) \end{Bmatrix}$$

On solving, we have

$$d^3(Sw,z) \leq \psi \begin{Bmatrix} d^2(z,Sw)d(z,z), \\ d(z,Sw)d^2(z,z) \\ d(z,Sw)d(z,z)d(z,Sw), \\ d(z,z)d(z,Sw)d(z,z) \end{Bmatrix}$$

This implies that $Sw = z$. Since the pair (S, A) is weakly commuting on X, therefore,

$$d(ASw, SAw) \leq d(Sw, Aw) = d(z,z) = 0, \text{ therefore,}$$

$$Az = Sz.$$

Hence $z = Az = Sz = Bz = Tz$.

Therefore, z is a common fixed point of S, T, A and B.

Case 4. Suppose that T is continuous, we can obtain a similar result by way of case 3. **Uniqueness**: Suppose $z \neq w$ be two common fixed points of S,T, A and B.

On putting $x = z$ and $y = w$ in (C2), we have

$d^3(Sz, Tw) \leq \psi\{0,0,0,0\}$

On solving we have $d^2(z, w) = 0$.

This implies $z = w$.

This completes the proof.

Pointwise R-Weakly Commuting & Reciprocal Continuous Mappings

Now we prove a common fixed point theorem using the notion of point wise R-weakly commuting mappings along with the notion of compatible and reciprocal continuous mappings.

Theorem 3.2 Let S,T, A and B be four mappings of a complete metric space (X, d) into itself satisfying (C1), (C2) and the following conditions:

(3.6) (A, S) and (B, T) are point wise R-weakly commuting pairs, (3.7) (A, S) and (B, T) are compatible pairs of reciprocally continuous mappings.

Then S,T, A and B have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. From (C1), we can find x_1 such that $S(x_0) = B(x_1) = y_0$. For this, x_1 one can find $x_2 \in X$ such that $T(x_1) = A(x_2) = y_1$. Continuing in this way, one can construct a sequence $\{y_n\}$ such that

$y_{2n} = S(x_{2n}) = B(x_{2n+1}), y_{2n+1} = T(x_{2n+1}) = A(x_{2n+2})$ for each $n \geq 0$.

From the proof of Theorem 3.1, $\{y_n\}$ is a Cauchy sequence in X.

From the completeness of X, there exists a $z \in X$ such that $y_n \to z$ as $n \to \infty$.

Moreover, since

$y_{2n+1} = T(x_{2n+1}) = A(x_{2n+2})$ and $y_{2n} = S(x_{2n}) = B(x_{2n+1})$ are subsequences of $\{y_n\}$, we obtain

$$\lim_{n\to\infty} T(x_{2n+1}) = \lim_{n\to\infty} A(x_{2n+2}) = \lim_{n\to\infty} S(x_{2n}) = \lim_{n\to\infty} B(x_{2n+1}) = z.$$

If B and T are compatible, then

$$\lim_{n\to\infty} d(BTx_n, TBx_n) = 0\,;$$

that is, $Bz = Tz$. Also by the reciprocal continuity of B and T, we have

$\lim_{n\to\infty} BTx_{2n} = Bz$ and $\lim_{n\to\infty} TBx_{2n} = Tz$.

Since $T(X) \subset A(X)$, there exists a point w in X such that $Tz = Aw$.

Setting $x = w$ and $y = z$ in (C2), we get

$$d^3(Sw, Tz) \leq \psi \begin{Bmatrix} d^2(Aw, Sw)d(Bz, Tz) \\ , d(Aw, Sw)d^2(Bz, Tz), \\ d(Aw, Sw)d(Aw, Tz)d(Bz, Sw), \\ d(Aw, Tz)d(Bz, Sw)d(Bz, Tz) \end{Bmatrix}$$

This implies that

$$d^3(Sw, Tz) \leq \psi \begin{Bmatrix} d^2(Tz, Sw)d(Tz, Tz) \\ +d(Tz, Sw)d^2(Tz, Tz), \\ d(Tz, Sw)d(Tz, Tz)d(Tz, Sw), \\ d(Tz, Tz)d(Tz, Sw)d(Tz, Tz) \end{Bmatrix},$$

i.e., $d^3(Sw, Tz) \leq \psi\{0,0,0,0\}$, using the properties of ψ, we have $Sw = Tz$, and hence $Sw = Tz = Aw = Bz$.

The point wise R-weak commutativity of B and T implies that there exists an $R > 0$ such that

$$d(BTz, TBz) \leq Rd(Bz, Tz),$$

which implies that $BTz = TBz$ and $TTz = TBz = BTz = BBz$.

Similarly, the point wise R-weak commutativity of A and S implies that there exists an $R > 0$ such that

$$d(ASw, SAw) \leq Rd(Aw, Sw),$$

which implies that $ASw = SAw$ and $AAw = ASw = SAw = SSw$.

Again substituting $x = w$ and $y = Tz$ in (C2), we get

$$d^3(Sw, TTz) \leq \psi \begin{Bmatrix} d^2(Aw, Sw)d(BTz, TTz) \\ d(Aw, Sw)d^2(BTz, TTz), \\ d(Aw, Sw)d(Aw, TTz)d(BTz, Sw), \\ d(Aw, TTz)d(BTz, Sw)d(BTz, TTz) \end{Bmatrix}$$

On simplification and using the properties of ψ, we have

$$d^3(Tz, TTz) \leq \psi\{0,0,0,0\}$$

Hence $Tz = TTz$. Thus $Tz = TTz = BTz$.

Therefore, Tz is a common fixed point of B and T.

Taking $x = Sw$ and $y = z$ in (C2), we get

$$d^3(SSw, Tz) \leq \psi \begin{Bmatrix} d^2(ASw, SSw)d(Bz, Tz), \\ d(ASw, SSw)d^2(Bz, Tz), \\ d(ASw, SSw)d(ASw, Tz)d(Bz, SSw), \\ d(ASw, Tz)d(Bz, SSw)d(Bz, Tz) \end{Bmatrix}.$$

On solving, we have

$$d^3(SSw, Sw) \leq \psi\{0,0,0,0\}$$

Using the properties of ψ, we have

Hence $Sw = SSw$. Thus $Sw = SSw = AAw$,

Thus Sw is a common fixed point of A and S.

If $Sw = Tz = u$, then $Tu = Bu = Su = Au = u$. Hence u is a common fixed point of A, B, S and T.

Uniqueness: Suppose that $v \neq u$ are two common fixed points of S, T, A and B.

On putting $x = u$ and $y = v$ in (C2), we have

$$d^3(Su,Tv) \leq \psi\{0,0,0\}$$

i.e., $d^3(u,v) \leq \psi\{0,0,0\}$

$i.e.,\ d^2(u,v) = 0$, This implies $u = v$.

This completes the proof.

R- Weakly Commuting Mappings of Type (P).

In 2009, Kumar et al. [17] defined the concept of R- weakly commuting mappings of type (P) in metric spaces. Now, we prove a common fixed point theorem for pairs of R- weakly commuting mappings of type (P) .

Theorem 3.3 Let S,T, A and B be four mappings of a complete metric space (X,d) into itself satisfying (C1), (C2),(3.1) and the following conditions:

(3.8) (A,S) and (B,T) are R- weakly commuting of type (P),

Then S,T, A and B have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. From (C2) we can find an x_1 such that $S(x_0) = B(x_1) = y_0$. For this x_1 one can find an $x_2 \in X$ such that $T(x_1) = A(x_2) = y_1$. Continuing in this way, one can construct a sequence such that

$y_{2n} = S(x_{2n}) = B(x_{2n+1}), y_{2n+1} = T(x_{2n+1}) = A(x_{2n+2})$ for each $n \geq 0$.

From the proof of Theorem 3.1, $\{y_n\}$ is a Cauchy sequence in X.

By the completeness of X, there exists a $z \in X$ such that $y_n \to z$ as $n \to \infty$.

Moreover, since

$y_{2n+1} = T(x_{2n+1}) = A(x_{2n+2})$ and $y_{2n} = S(x_{2n}) = B(x_{2n+1})$ are subsequences of $\{y_n\}$, so these subsequences

$T(x_{2n+1}) = A(x_{2n+2}) = S(x_{2n}) = B(x_{2n+1})$ also converges to the same limit as $n \to \infty$.

Case 1: Suppose that A is continuous. Then $\{AAx_{2n}\}$ and $\{ASx_{2n}\}$ converges to Az as $n \to \infty$. Since the mappings A and S are R-weakly commuting of type (P), we have

$$d(SSx_{2n}, AAx_{2n}) \leq Rd(Ax_{2n}, Sx_{2n}).$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} SAx_{2n} = Az$.

On putting $x = Ax_{2n}$ and $y = x_{2n+1}$ in (C2), we get

$$d^3(SAx_{2n}, Tx_{2n+1}) \leq \psi \left\{ \begin{array}{c} d^2(AAx_{2n}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \\ , d(AAx_{2n}, SAx_{2n})d^2(Bx_{2n+1}, Tx_{2n+1}), \\ d(AAx_{2n}, SAx_{2n})d(AAx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SAx_{2n}), \\ d(AAx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \end{array} \right\}.$$

Proceeding limit as $n \to \infty$, we get

$d^3(Az, z) \leq \psi\{0,0,0,0\}$

i.e., $d^3(Az, z) \leq 0$. Using the properties of ψ, we have $d^2(Az, z) = 0$, i.e., $Az = z$.

Next, we shall show that $Sz = z$.

For this, putting $x = z$ and $y = x_{2n+1}$ in (C2), we get

$$d^3(Sz, Tx_{2n+1}) \leq \psi \left\{ \begin{array}{c} d^2(Az, Sz)d(z, z) \\ , d(Az, Sz)d^2(z, z), \\ d(Az, Sz)d(Az, z)d(z, Sz), \\ d(Az, z)d(z, Sz)d(z, z) \end{array} \right\}$$

Therefore,

$d^3(Sz, z) \leq \psi\{0,0,0,0\}$. Using the properties of ψ , we have

Thus, $d^2(Sz, z) = 0$, implies $Sz = z$.

Since $S(X) \subset B(X)$, therefore, there exists a point $u \in X$ such that $z = Sz = Bu$.

We claim that $z = Tu$.

For this, on putting $x = z$ and $y = u$ in (C2), we get

$$d^3(Sz, Tu) \leq \psi \left\{ \begin{array}{c} d^2(Az, Sz)d(Bu, Tu), \\ d(Az, Sz)d^2(Bu, Tu) \\ d(Az, Sz)d(Az, Tu)d(Bu, Sz), \\ d(Az, Tu)d(Bu, Sz)d(Bu, Tu) \end{array} \right\},$$

Thus we have

$$d^3(z, Tu) \leq \psi \left\{ \begin{array}{c} d^2(z, z)d(z, Tu), \\ d(z, z)d^2(z, Tu) \\ d(z, z)d(z, Tu)d(z, z), \\ d(z, Tu)d(z, z)d(z, Tu) \end{array} \right\}$$

Using the properties of ψ, we have $z = Tu$. Since (B, T) is R-weakly commuting of type (P), we have $d(Bz, Tz) = d(BBu, TTu) \leq Rd(Tu, Bu) = Rd(z, z) = 0$.

Hence $Bz = Tz$.

Finally, we have

$$d^3(Sz,Tz) \leq \psi\begin{Bmatrix} d^2(Az,Sz)d(Bz,Tz), \\ d(Az,Sz)d^2(Bz,Tz), \\ d(Az,Sz)d(Az,Tz)d(Bz,Sz), \\ d(Az,Tz)d(Bz,Sz)d(Bz,Tz) \end{Bmatrix}$$

On simplification, we have

$$d^2(z,Tz) \leq \psi\{0,0,0,0\}$$

Using the properties of ψ, we have $z = Tz$. Hence $z = Bz = Tz = Az = Sz$. Therefore, z is a common fixed point of S, T, A and B.

Case 2: Suppose that B is continuous. Then we can obtain the same result by using Case 1.

Case 3: Suppose that S is continuous.

Then $\{SSx_{2n}\}$ and $\{SAx_{2n}\}$ converge to Sz as $n \to \infty$.

Since the mappings A and S are R-weakly commuting of type (P), we have $d(AAx_{2n}, SSx_{2n}) \leq Rd(Sx_{2n}, Ax_{2n})$.

Letting $n \to \infty$, we get $\lim_{n\to\infty} ASx_{2n} = Sz$.

On putting $x = Sx_{2n}$ and $y = x_{2n+1}$ in (C2), we get

$$d^3(SSx_{2n}, Tx_{2n+1}) \leq \psi\begin{Bmatrix} d^2(ASx_{2n}, SSx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \\ , d(ASx_{2n}, SSx_{2n})d^2(Bx_{2n+1}, Tx_{2n+1}), \\ d(ASx_{2n}, SSx_{2n})d(ASx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SSx_{2n}), \\ d(ASx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SSx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \end{Bmatrix}$$

Proceeding limit as $n \to \infty$, we get

$$d^3(Sz,z) \leq \psi\{0,0,0,0\}$$

i.e., $d^3(Sz,z) \leq 0$

Thus we get $d^3(Sz,z) = 0$, which implies that $Sz = z$.

Since $S(X) \subset B(X)$, there exists a point $v \in X$ such that $z = Sz = Bv$.

We claim that $z = Tv$.

For this, putting $x = Sx_{2n}$ and $y = v$ in (C2), we get

$$d^2(SSx_{2n},Tv) \leq \psi\begin{Bmatrix} \frac{1}{2}\begin{bmatrix} d^2(ASx_{2n},SSx_{2n})d(Bv,Tv) \\ +d(ASx_{2n},SSx_{2n})d^2(Bv,Tv) \end{bmatrix}, \\ d(ASx_{2n},SSx_{2n})d(ASx_{2n},Tv)d(Bv,SSx_{2n}), \\ d(ASx_{2n},Tv)d(Bv,SSx_{2n})d(Bv,Tv) \end{Bmatrix}.$$

On simplification, we get

$$d^3(z,Tv) \leq \psi\begin{Bmatrix} d^2(z,z)d(z,Tv), \\ d(z,z)d^2(z,Tv) \\ d(z,z)d(z,Tv)d(z,z), \\ d(z,Tv)d(z,z)d(z,Tv) \end{Bmatrix}$$

Using the properties of ψ , we have $z = Tv$.

Since (B,T) is R-weakly commuting of type (P),

we have $d(Tz,Bz) = d(TTv,BBv) \leq Rd(Bv,Tv) = Rd(z,z) = 0$.

This gives $Bz = Tz,$ for R > 0.

Finally, from (C2) we have

$$d^3(Sx_{2n},Tz) \leq \psi\begin{Bmatrix} d^2(Ax_{2n},Sx_{2n})d(Bz,Tz), \\ d(Ax_{2n},Sx_{2n})d^2(Bz,Tz), \\ d(Ax_{2n},Sx_{2n})d(Ax_{2n},Tz)d(Bz,Sx_{2n}), \\ d(Ax_{2n},Tz)d(Bz,Sx_{2n})d(Bz,Tz) \end{Bmatrix}$$

Therefore, we have

$d^3(z,Tz) \leq \psi\{0,0,0,0\}$. Using the properties of ψ, we have

$z = Tz$.

Since $T(X) \subset A(X)$, therefore, there exists a point $w \in X$ such that $z = Tz = Aw$.

We claim that $z = Sw$.

To establish this, on putting $x = w$ and $y = z$ in (C2), we get

$$d^3(Sw,Tz) \leq \psi\begin{Bmatrix} d^2(Aw,Sw)d(Bz,Tz), \\ d(Aw,Sw)d^2(Bz,Tz) \\ d(Aw,Sw)d(Aw,Tz)d(Bz,Sw), \\ d(Aw,Tz)d(Bz,Sw)d(Bz,Tz) \end{Bmatrix}$$

Hence we get

$$d^3(Sw,z) \leq \psi \left\{ \begin{matrix} d^2(z,Sw)d(z,z), \\ d(z,Sw)d^2(z,z) \\ d(z,Sw)d(z,z)d(z,Sw), \\ d(z,z)d(z,Sw)d(z,z) \end{matrix} \right\}$$

Properties of ψ, implies that $Sw = z$.

Since (S,A) is R-weakly commuting of type (P), we have $d(Az,Sz) = d(AAw,SSw) \leq Rd(Sw,Aw) = Rd(z,z) = 0$. Hence $Az = Sz$.

Hence $z = Az = Sz = Bz = Tz$, and z is a common fixed point of S,T,A and B.

Case 4: Suppose that T is continuous. We can obtain the same result by using Case 3.

Uniqueness: Suppose that $z \neq w$ are two common fixed points of S,T,A and B.

On putting $x = z$ and $y = w$ in (C2), we get

$$d^3(Sz,Tw) \leq \psi\{0,0,0,0\}$$

$$i.e., d^2(z,w) = 0 \text{ implies } z = w.$$

This completes the proof.

Example 3.1 Let $X = [2,20]$ and d be a usual metric. Define the self mappings A,B,S and T on X by

$$Ax = \begin{cases} 12 & if \quad 2 < x \leq 5 \\ x-3 & if \quad x > 5 \\ 2 & if \quad x = 2. \end{cases}, \qquad Bx = \begin{cases} 2 & if \quad x = 2 \\ 6 & if \quad x > 2 \end{cases},$$

$$Sx = \begin{cases} 6 & if \quad 2 < x \leq 5 \\ x & if \quad x = 2 \\ 2 & if \quad x > 5. \end{cases} \quad \text{and} \quad Tx = \begin{cases} x & if \quad x = 2 \\ 3 & if \quad x > 2 \end{cases}.$$

Let us consider a sequence $\{x_n\}$ with $x_n = 2$. It is easy to verify that all the conditions of Theorem 3.1 are satisfied. In fact, 2 is the unique common fixed point of S,T,A and B.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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