Existence of Common Fixed Point Theorems for Conditionally Reciprocally Continuous Mappings in G- Metric Space

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Abstract: In this paper, our aim is to establish and prove some common fixed point theorems for a pair of conditionally reciprocally continuous mappings in the setting of G – metric spaces. Our results unify and extend certain fixed points results present in the literature.

Key Words: Common fixed point, G-metric space, Conditionally reciprocally continuity.

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1 Introduction

Fixed point theory is one of the most fruitful and effective tool in mathematics which has many application within as well as outside mathematics. The study of fixed point theory in metric space has been at the centre of vigorous activity and it has a wide range of application as said above in the applied mathematics and science. Over the last few decades a considerable amount of research work for the development of fixed point theory have executed by several researcher. Different generalizations of the usual notion of a metric space have been proposed by many authors.

In 2006, Mustafa and Sims [12] introduced a new notion of metric spaces, called G-metric spaces. After then, many authors studied fixed point in G-metric spaces. Some of these work may be noted in [13-16] and in [3-8].

The main aim of this paper is to established coincidence and common fixed point theorems for maps satisfying the conditions of faint compatibility and conditional reciprocal continuity.

2. Preliminaries:

Definition 1.1. [12] G-metric spaces

In 2006, Mustafa and Sims introduced the concept of G-metric space as follows:

Let X be a nonempty set, and $let G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following:

(G1) $G(x, y, z) = 0 \, if x = y = z$,

(G2) $0 < G(x, x, y) \, for all x, y in X with x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z) \, for all x, y, z in X with z \neq y$,

$(G4)$ $(x, z, y) = (G(x, y, z) = G(y, z, x) = \ldots$ (symmetry in all three variables),

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z) \, for all x, y, z, a in X$ (rectangle inequality).

Then the function G is called a generalized metric or, more specifically, a G-metric on X and the pair (X, G) is called a G-metric space.

Definition 1.2.[12] Let(X, G) be a $G -$ metric space and $\{x_n\}$ be a sequence of points in X. We can say that $\{x_n\}$ is $G -$ convergent to x if $\lim_{n \to \infty} G(x, x_n, x_m) = 0$, this implies that for each $\epsilon > 0$ there exists a positive integer N such that $G(x, x_n, x_n) < \epsilon \forall m, n \geq N$. We can say that x is the limit of the sequence and can write $x_n \to x$ or $\lim_{n \to \infty} x_n = x$.

Proposition 1.1.[12] Let (X, G) be a $G -$metric space then the following are equivalent:

1. $\{x_n\}$ is $G -$ convergent to x,
2. $G(x_n, x_n, x) \to 0 \, as n \to \infty$,
3. $G(x_n, x, x) \to 0 \, as n \to \infty$,
4. $G(x_m, x_n, x) \to 0 \, as n \to \infty$.

Definition 1.3.[12] Let (X, G) be a $G -$ metric space. A sequence $\{x_n\}$ is said to be a $G -$ Cauchy sequence for each $\epsilon > 0$ then there exists a positive integer N such that $G(x_m, x_n, x_l) < \epsilon \forall l, m, n \geq N$.

Proposition 1.2. [12] let (X, G) be a $G -$ metric space then the following are equivalent :

1.The sequence $\{x_n\}$ is $G -$ Cauchy,

2.For each $\epsilon > 0$ there exists a positive integer N such that $G(x_m, x_n, x_l) < \epsilon \forall l, m, n \geq N$.

Definition 1.4. [12] A $G -$ metric space $(X, G)$is called a symmetric $G -$ metric if $G(x, y, y) = G(y, x, x) \forall x, y \in X$.

Proposition 1.3. [12] A $G -$ metric space $(X, G)$is called a $G -$ complete if and only if (X, d_G) is a complete metric space.

Proposition1.4.[12]] Let (X, G) be a $G -$ metric space .Then, for any $x, y, z, a \in X$ it follows that

1. If$G(x, y, z) = 0$ then $x = y = z$,
2. $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
3. $G(x, y, y) \leq 2G(y, xx)$,
4. $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,
5. $G(x, y, z) \leq \frac{2}{3}\big(G(x, a, a) + G(y, a, a) + G(z, a, a)\big)$.

Now we give example of non symmetric G-metric spaces.

Example1.1. [12]$Let X = \{a,b\}$, and G$(a,a,a) = G(a,a,a) = 0, G(a,a,b) = 1$, $G(a,b,b) = 2$ and extend G to all of $X \times X \times X$ by symmetry in the variables. Then$G$ is a G-metric space. It is non symmetric since $G(a,b,b) \neq G(a,a,b)$.

Definition 1.5. [1] Let $fandg$ be two self mappings on G- metric space (X,G). The mappings $fandg$ are said to be compatible if

$\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

Example 1.2.[1]Let $X = [0,3]$ and let$G$ be a $G-$ metric on $X \times X \times X$ defined as follows:$G(x,y,z) = |x-y| + |y-z| + |z-x| \forall x,y,z \in X$.Now f,g are defined as follows:$fx = \begin{cases} 0 & if\ x \in [0,1), \\ 3 & if\ x \in [1,3] \end{cases}$and $gx = \begin{cases} 3-x & if\ x \in [0,1) \\ 3 & if\ x \in [1,3] \end{cases}$Then for any $x \in [1,3]$, $x$is a coincidence point and $fgx = gfx$, showing that f,g are compatible mappings.

Definition 1.6. [2] Two maps $fandg$ are said to be weakly compatible if they commute at coincidence points.

Definition 1.7. [1] Let f and g be two self mappings of a G- metric space (X,G) . Then the pair (f,g) is said to be satisfy the property E.A. if there exists a sequence (x_n) in X such that

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Definition 1.8.[2] A pair of self mappings (f,g) of a G-metric space (X,G) is said to be non compatible if there exists a sequence (x_n) in X such that

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.But $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n)$ is either non zero or non – existent.

Definition 1.9.[10]Two self mappings $fandg$ of a $G-$ metric space (X,G) are said to be faintly compatible iff $fandg$ are conditionally compatible and they commute on a nonempty subset of coincidence points whenever the set of coincidences is nonempty.

Definition 1.10.[9] Two self mappings $fandg$ are said to be conditionally reciprocally continuous if, whenever the set of sequences $\{x_n\}$ in X satisfying

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$ is nonempty, there exit a sequence $\{y_n\}$ in X satisfying $\lim_{n\to\infty} f y_n = \lim_{n\to\infty} gy_n = t$ (say) such that

$$\lim_{n\to\infty} fgy_n = ft and \lim_{n\to\infty} gfy_n = gt.$$

Definition 1.11.[9] A pair of self mappings$fandg$ is said to be reciprocally continuous iff $\lim_{n\to\infty} fgx_n = ft$, $\lim_{n\to\infty} gx_n = gt$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n =$

$\lim_{n\to\infty} gx_n = t$for some $tinX$. Then $AandB$ have a coincidence point

If $fandg$ are both continuous, then then they are reciprocally continuous but the converse need not be true.

Definition 1.12.[11]A pair of self mappings $fandg$ is said to be conditionally compatible if whenever $\{x_n\}$ is a sequence in X such that

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ is nonempty, there exist a sequence $\{y_n\}$ in X satisfying$\lim_{n\to\infty} f\, y_n = \lim_{n\to\infty} gy_n = t$ (say) and $\lim_{n\to\infty} G\,(fgy_n, gfy_n, gfy_n) = 0$.

Definition 1.13.[9]The pair of self mappings $fandg$ is said to fulfill the condition of (CLRg) property, if there exist a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$for some $xinX$.

2. Main Results:

Theorem 2.1. Let $AandB$ be a pair of self mappings of a symmetric G- metric space (X, G) such that

2.1.$AandB$ satisfy the property $(E.A)$.

2.2.$AandB$ are conditionally reciprocally continuous mappings.

2.3. $AandB$are faintly compatible mappings.

Then $AandB$ have a coincidence point. Moreover$AandB$ have a unique common fixed point if the pair satisfies the following inequality.

2.4. $G(Ax, Ay, Az)$

$\leq \lambda max\{G(Bx, By, Bz), G(Ax, Bx, Bx), G(Ay, By, Bz), G(Ax, By, Bz), G(Ay, Bx, Bx)\}$,

$0 \leq \lambda < 1$.

Proof. The mappings $AandB$ satisfy the property $(E.A)$. Then there exists a sequence $\{x_n\}$ of points of X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$for some $tinX$.

Also the two mappings $AandB$ are faintly compatible .So there exist a sequence $\{y_n\}$ in X satisfying $\lim_{n\to\infty} A\, y_n = \lim_{n\to\infty} By_n = v$ (say) and $\lim_{n\to\infty} G\,(ABy_n, BAy_n, BAy_n) = 0$.

Since$AandB$ are conditionally reciprocally continuous mappings. Then we have $\lim_{n\to\infty} ABy_n = Av$and $\lim_{n\to\infty} BAy_n = Bv$. This implies $Av = Bv$. In other words mappings $AandB$ have a coincidence point.

Also $AandB$are faintly compatible mappings. Then we have $ABv = BAv$.

Hence $ABv = BAv = AAv = BBv$.

Now we have to prove that$Av = AAv$. If it does not happen, then using inequality (4)

$$G(Av, AAv, AAv) \leq \lambda max\left\{\begin{matrix} G(Bv, BAv, BAv), G(Av, Bv, Bv), G(AAv, BAv, BAv) \\ , G(Av, BAv, BAv), G(AAv, Bv, Bv) \end{matrix}\right\}$$

Since$ABv = BAv = AAv = BBv$. So we have,

$$G(Av, AAv, AAv) \leq \lambda max \begin{Bmatrix} G(Av, AAv, AAv), G(Av, Av, Av), G(AAv, AAv, AAv) \\ , G(Av, AAv, AAv), G(AAv, Av, Av) \end{Bmatrix}$$

$$G(Av, AAv, AAv) \leq \lambda max \begin{Bmatrix} G(Av, AAv, AAv), 0,0 \\ , G(Av, AAv, AAv), G(AAv, Av, Av) \end{Bmatrix}$$

Since G is symmetric G- metric space so $G(Av, AAv, AAv) = G(AAv, Av, Av)$.

Then $G(Av, AAv, AAv) \leq \lambda max\{G(AAv, Av, Av)\}$.

A contradiction, so our supposition is wrong. Hence $Av = AAv$. This implies that Av is a common fixed point of the mappings $andB$.

For uniqueness. Let Aw be another common fixed point of the mappings $AandB$.Then $Aw \neq Av$. This implies that $G(Av, Aw, Aw) > 0$, where w is the coincidence point of the pair of mappings $AandB$. Now using inequality (2.4) we have$G(Av, Aw, Aw) \leq$

$$\lambda max \begin{Bmatrix} G(Bv, Bw, Bw), G(Av, Bv, Bv), G(Aw, Bw, Bw) \\ , G(Av, Bw, Bw), G(Aw, Bv, Bv) \end{Bmatrix}$$

Since w and v are the coincidence points of the pair of mappings $AandB$, then $Aw = Bw$ and $Av = Bv$, using these values in above inequality we have.

$$G(Av, Aw, Aw) \leq \lambda max \begin{Bmatrix} G(Av, Aw, Aw), G(Av, Av, Av), G(Aw, Aw, Aw) \\ , G(Av, Aw, Aw), G(Aw, Av, Av) \end{Bmatrix}$$

$$G(Av, Aw, Aw) \leq \lambda max \begin{Bmatrix} G(Av, Aw, Aw), 0,0 \\ , G(Av, Aw, Aw), G(Aw, Av, Av) \end{Bmatrix}$$

Since G is symmetric G- metric space so $G(Av, Aw, Aw) = G(Aw, Av, Av)$. So we have $G(Av, Aw, Aw) \leq \lambda max\{G(Av, Aw, Aw)\}$. Which is a contradiction. Hence $Aw = Av$ is a unique common fixed point of the pair of mappings $AandB$.

Theorem 2.2. Let $AandB$ be a pair of self mappings of a symmetric G- metric space (X, G) is satisfying properties (2.1), (2.2), (2.3). Then $AandB$ have a coincidence point. Moreover $AandB$ have a unique common fixed point if the pair satisfies the following inequality.

2.5. $G(Ax, Ay, Az)$
$< \lambda max\{G(Bx, By, Bz), G(Ax, Bx, Bx), G(Ay, By, Bz), G(Ax, By, Bz), G(Ay, Bx, Bx)\}$,
$0 \leq \lambda < 1$.

Proof. Proof of the theorem follows on the similar lines as of Theorem 2.1.

Theorem 2.3. Let $AandB$ be a pair of self mappings of a symmetric G- metric space (X, G) is satisfying the property (2.1), (2.2), (2.3).Then $AandB$ have a coincidence point. Moreover $AandB$ have a unique common fixed point if the pair satisfies the following inequality.

2.6. $G(Bx, BBx, BBy) \neq max\{G(Bx, ABx, ABy), G(ABx, BBx, BBy)\}$.

Proof. On the similar lines as in Theorem 2.1 we can prove that

$ABv = BAv = AAv = BBv$.

Now we prove that $BBv = Bv$. If this does not happen , then by inequality (2.6) we have.
$G(Bv, BBv, BBv) \neq max\{G(Bv, ABv, ABv), G(ABv, BBv, BBv)\}$.
$G(Bv, BBv, BBv) \neq max\{G(Bv, BBv, BBv), G(ABv, AB, ABv)\}$.
$G(Bv, BBv, BBv) \neq max\{G(Bv, BBv, BBv)\}$. Which is a contradiction . Hence$BBv = Bv$. This implies that Bv is a common fixed point of the pair of mappings $AandB$.Uniqueness follows from the Theorem 2.1.

Theorem 2.4. Let $AandB$ be a pair of self mappings of a symmetric G- metric space (X, G)satisfying the inequality (2.4) and

2.7.Weak compatibility.

2.8.The common limit in the range of g property that is (CLRg) property.

Then $AandB$ have a coincidence point and unique common fixed point.

Proof. Since the pair of self mappings $AandB$ fulfill the condition of (CLRg) property, then there exist a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = Bx$for some $xinX$. Put $x = x_n, y = x, z = x$ in inequality (2.4) we have.

$$G(Ax_n, Ax, Ax) \leq \lambda max\begin{Bmatrix} G(Bx_n, Bx, Bx), G(Ax_n, Bx_n, Bx_n), G(Ax, Bx, Bx), \\ G(Ax_n, Bx, Bx), G(Ax, Bx_n, Bx_n) \end{Bmatrix}.$$

Taking limit as$n \to \infty$, we get.

$$G(Bx, Ax, Ax) \leq \lambda max\begin{Bmatrix} G(Bx, Bx, Bx), G(Bx, Bx, Bx), G(Ax, Bx, Bx), \\ G(Bx, Bx, Bx), G(Ax, Bx, Bx) \end{Bmatrix}.$$

$$G(Bx, Ax, Ax) \leq \lambda max\begin{Bmatrix} 0, 0, G(Ax, Bx, Bx), \\ 0, G(Ax, Bx, Bx) \end{Bmatrix}.$$

$G(Bx, Ax, Ax) \leq \lambda max\{G(Ax, Bx, Bx)\}$. Since G is symmetric G- metric space so $G(Bx, Ax, Ax) = G(Ax, Bx, Bx)$.This implies that
$G(Bx, Ax, Ax) \leq \lambda max\{G(Bx, Ax, Ax)\}$. Which is a contradiction.

Hence $Ax = Bx$. That is the two mappings $AandB$ have a coincidence point. Let $Ax = Bx = z$. Since the two mappings $AandB$ satisfy the property of weak compatibility. So we have $Az = ABx = BAx = Bz$. Now we claim that $Az = z$. If this does not happen, then using the inequality (2.4) we get.

$$G(Az, z, z) = G(Az, Ax, Ax)$$
$$\leq \lambda max\{G(Bz, Bx, Bx), G(Az, Bz, Bz), G(Ax, Bx, Bx), G(Az, Bx, Bx), G(Ax, Bz, Bz)\}$$

Since $Ax = Bx = z$ and $Az = ABx = BAx = Bz$ So the above equation reduces to

$$G(Az, z, z) \leq \lambda max\{G(Az, z, z), G(Az, Az, Az), G(z, z, z), G(Az, z, z), G(zAz, Az)\}.$$

$G(Az, z, z) \leq \lambda max\{G(Az, z, z), 0,0, G(Az, z, z), G(zAz, Az)\}$.

Since G is symmetric G- metric space so $G(Az, z, z) = G(z, Az, Az)$. So we have ,$G(Az, z, z) \leq \lambda max\{G(Az, z, z)\}$.Which is a contradiction.

Hence $Az = Bz = z$. This implies that z is a common fixed point of the two mappings $AandB$.
For uniqueness of common fixed point is an easy consequence of the condition (2.4).

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