

# Existence and uniqueness of fixed point for Meir-Keeler type contractive condition in Menger spaces

Vishal Gupta<sup>1</sup>  | Mohammad S. Khan<sup>2</sup> | Balbir Singh<sup>3</sup> | Sanjay Kumar<sup>4</sup>

<sup>1</sup>Department of Mathematics, Maharishi Markandeshwar (Deemed to be University), Mullana, India

<sup>2</sup>Department of Mathematics and Statistics, Sultan Qaboos University, Muscat, Sultanate of Oman

<sup>3</sup>School of Physical Sciences, Starex University, Gurugram, India

<sup>4</sup>Department of Mathematics, D.C.R. University of Science and Technology, Sonapat, India

## Correspondence

Vishal Gupta, Department of Mathematics, Maharishi Markandeshwar (Deemed to be University) Mullana 133207, Haryana, India.  
Email: vishal.gmn@gmail.com; vgupta@mmumullana.org

## Abstract

In this paper, we prove some general common fixed point theorems using generalized contractive condition of Meir-Keeler type for two pairs of weakly compatible self-mappings in Menger spaces. Some suitable examples are also given to support our theorems.

## KEYWORDS

common property (E.A), compatible mappings,  $JCLR_{ST}$  property, Menger spaces, weakly compatible

## 1 | INTRODUCTION

Menger<sup>1</sup> introduced the notion of probabilistic metric spaces as a generalization of metric space. The notion of Probabilistic Metric space in the Menger theory refers to situations where we know the probabilities of possible values of the distance, but we do not know precisely the distance between two points. Menger explained in his note how the numerical distance between two points  $x$  and  $y$  could be replaced by a function  $\mathcal{F}(x, y, t)$ , whose value  $\mathcal{F}(x, y, t)$  is interpreted at the real number  $t$  as the probability that the distance between  $x$  and  $y$  is less than  $t$ . In fact the analysis of these spaces got an impetus with Schweizer and Sklar's pioneering work.<sup>2</sup> In probabilistic functional analysis the theory of probabilistic metric space is of paramount importance particularly due to its extensive applications in random differential as well as random integral equations.

**Definition 1.** (2) A distribution function  $\mathcal{J} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a left continuous and non-decreasing function with  $\inf\{\mathcal{J}(u) : u \in \mathbb{R}^+\} = 0$  and  $\sup\{\mathcal{J}(u) : u \in \mathbb{R}^+\} = 1$ .  $\mathfrak{F}$  is the set of all distribution functions and  $\mathcal{H}$  be the Heaviside function defined by  $\mathcal{H}(t) = \begin{cases} 1, & t > 0 \\ 0, & t \leq 0. \end{cases}$

**Definition 2.** (1) A pair  $(\mathfrak{D}, \mathcal{F})$  is a PM-space, where  $\mathfrak{D}$  is a nonempty set and  $\mathcal{F} : \mathfrak{D} \times \mathfrak{D} \times [0, 1] \rightarrow \mathfrak{F}$  is a mapping satisfying the following properties for all  $u, v, w \in \mathfrak{D}$  and  $t, s \geq 0$ ,

- (p<sub>1</sub>)  $\mathcal{F}(u, v, t) = 1$  iff  $u = v$ ;
- (p<sub>2</sub>)  $\mathcal{F}(u, v, 0) = 0$ ;
- (p<sub>3</sub>)  $\mathcal{F}(u, v, t) = \mathcal{F}(v, u, t)$ ;
- (p<sub>4</sub>)  $\mathcal{F}(u, v, t) = 1$  and  $\mathcal{F}(v, w, s) = 1$ , then  $\mathcal{F}(u, w, (t + s)) = 1$ .

Every metric space  $(\mathfrak{D}, d)$  can always be realized as a PM-space by  $F(p, q, t) = H(t - d(p, q)), \forall p, q \in \mathfrak{D}$ , and  $H$  be the Heaviside function defined by

$$H(t) = \begin{cases} 1, & t > 0 \\ 0, & t \leq 0 \end{cases} \text{ and } F: \mathfrak{D} \times \mathfrak{D} \rightarrow \mathfrak{I}.$$

Probabilistic Metric space has a broader context than the metric space, which encompasses much broader statistical situations.

**Definition 3.** (2) A mapping  $\Delta: [0, 1] \times [0, 1] \rightarrow [0, 1]$  is called a  $t$ -norm if for all  $a, b, c \in [0, 1]$ ,

- (1)  $\Delta(a, 1) = a, \Delta(0, 0) = 0$ ;
- (2)  $\Delta(a, b) = \Delta(b, a)$ ;
- (3)  $\Delta(c, d) \geq \Delta(a, b)$  for  $c \geq a, d \geq b$ ;
- (4)  $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$ .

**Example 1.** The four basic  $t$ -norms are minimum  $t$ -norm, product  $t$ -norm, Lukasiewicz  $t$ -norm and weakest  $t$ -norm, the drastic product.

**Definition 4.** (1) A Menger space is a triplet  $(\mathfrak{D}, F, \Delta)$ , where  $(\mathfrak{D}, F)$  is a PM-space and  $\Delta$  is a  $t$ -norm satisfying the property,

$$(p_5) F(u, w, (t + s)) \geq \Delta(F(u, v, t), F(v, w, s)), \forall u, v, w \in \mathfrak{D} \text{ and } t, s \geq 0,$$

**Example 2.** Let  $\mathfrak{D} = \mathbb{R}$ ,  $\Delta(a, b) = \min(a, b), \forall a, b \in [0, 1]$  and

$$F(u, v, t) = \begin{cases} H(t), & u \neq v \\ 1, & u = v \end{cases}; \quad \text{where } H(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0. \end{cases}$$

Then  $(\mathfrak{D}, F, \Delta)$  is a Menger space.

**Definition 5.** (3) A sequence  $\{x_n\}$  in Menger space  $(\mathfrak{D}, F, \Delta)$  is said to be:

- (i) convergent at a point  $u \in \mathfrak{D}$  if for every  $\epsilon > 0$  and  $\lambda > 0$ , there exists a positive integer  $N_{\epsilon, \lambda}$  s.t.  $F(x_n, u, \epsilon) > 1 - \lambda$ , for all  $n \geq N_{\epsilon, \lambda}$ .
- (ii) a Cauchy sequence in  $\mathfrak{D}$  if for every  $\epsilon > 0$  and  $\lambda > 0$ , there exists a positive integer  $N_{\epsilon, \lambda}$  s.t.  $F(x_n, x_m, \epsilon) > 1 - \lambda$ , for all  $n, m \geq N_{\epsilon, \lambda}$ .
- (iii) complete if every Cauchy sequence in  $\mathfrak{D}$  is convergent in  $\mathfrak{D}$ .

Weakly commuting mappings were introduced by Jungck in 1996.

**Definition 6.** (4) Two self-mappings  $f$  and  $g$  in a Menger space  $(\mathfrak{D}, F, \Delta)$  are weakly commuting if  $F(fgx, gfx, t) \geq F(fx, gx, t)$ , for all  $x \in \mathfrak{D}$  and  $t > 0$ .

Jungck<sup>5</sup> extended the Definition 6 to compatible mappings. In 1991, Mishra<sup>3</sup> introduced the notion of compatible mappings in the setting of PM-space.

**Definition 7.** (3) Two self-mappings  $f$  and  $g$  in a Menger space  $(\mathfrak{D}, F, \Delta)$  are said to be compatible if  $\lim_{n \rightarrow \infty} F(fgx_n, gfx_n, t) = 1$ , whenever  $\{x_n\}$  is a sequence in  $\mathfrak{D}$  such that  $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = w$  for some  $w \in \mathfrak{D}$  and for all  $t > 0$ .

**Definition 8.** Two self-mappings  $f$  and  $g$  in a Menger space  $(\mathfrak{D}, F, \Delta)$  are said to be non-compatible if either  $\lim_{n \rightarrow \infty} F(fgx_n, gfx_n, t)$  is non-existent or not equal to one, whenever  $\{x_n\}$  is a sequence in  $\mathfrak{D}$  such that  $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = w$  for some  $w \in \mathfrak{D}$  and for all  $t > 0$ .

**Definition 9.** (6) Self maps  $f$  and  $g$  of a Menger space  $(\mathfrak{D}, F, \Delta)$  are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, that is, if  $fx = gx$  for some  $x \in \mathfrak{D}$  then  $fgx = gfx$ . The concept of commuting mappings, weakly commuting mappings, weakly compatible mappings, compatible and non-compatible mappings in different other spaces has been introduced by the researchers [see, 7–12]. With the help of E.A, CLRg and

JCLR property many authors have proved common fixed point theorems in different spaces [see 13–15]. Also, by using Meir Keeler type contractions and  $\phi$ - $\psi$  type conditions various fixed point results are proved [see, 16–19].

**Definition 10.** (20) A pair of self-mappings  $(f, g)$  on a Menger space  $(\mathfrak{D}, \mathcal{F}, \Delta)$  is said to satisfy the property (E.A) if there exists a sequence  $\{u_n\}$  in  $\mathfrak{D}$  such that  $\lim_{n \rightarrow \infty} F(fu_n, u, t) = \lim_{n \rightarrow \infty} F(gu_n, u, t) = 1$ , for some  $u \in \mathfrak{D}$  and  $\forall t > 0$ .

**Definition 11.** The pairs  $(A, S)$  and  $(B, J)$  on a Menger space  $(\mathfrak{D}, \mathcal{F}, \Delta)$  are said to satisfy the common property (E.A) if there exist two sequences  $\{p_n\}$  and  $\{q_n\}$  in  $\mathfrak{D}$  s.t.

$$\lim_{n \rightarrow \infty} A p_n = \lim_{n \rightarrow \infty} S p_n = \lim_{n \rightarrow \infty} B q_n = \lim_{n \rightarrow \infty} J q_n = r, \text{ for some } r \in \mathfrak{D}.$$

Imdad<sup>21</sup> presented the  $JCLR_{ST}$  property in 2012.

**Definition 12.** (21) The pairs  $(A, S)$  and  $(B, J)$  on a Menger space  $(\mathfrak{D}, \mathcal{F}, \Delta)$  are said to satisfy the  $JCLR_{ST}$  property if there exist two sequences  $\{p_n\}$  and  $\{q_n\}$  in  $\mathfrak{D}$  s.t.

$$\lim_{n \rightarrow \infty} F(A p_n, u, t) = \lim_{n \rightarrow \infty} F(S p_n, u, t) = \lim_{n \rightarrow \infty} F(B q_n, u, t) = \lim_{n \rightarrow \infty} F(J q_n, u, t) = 1, \text{ where } u = S z = T z,$$

for some  $z \in \mathfrak{D}$ .

**Example 3.** Let  $(\mathfrak{D}, \mathcal{F}, \Delta)$  be a Menger space with  $\mathfrak{D} = [-1, 1]$  and  $F(p, q, t) = \frac{t}{t+|p-q|}$  for all  $p, q \in \mathfrak{D}, t > 0$  and  $F(p, q, 0) = 0$ , where  $\Delta(a, b) = \min\{a, b\}$  for all  $a, b \in [0, 1]$ . Define  $A, B, S$  and  $T$  self maps on  $\mathfrak{D}$  as  $A p = \frac{p}{5}, B p = \frac{-p}{5}, S p = p, T p = -p$  for all  $p \in \mathfrak{D}$ . Then with sequences  $\{p_n\} = \left\{\frac{1}{3n}\right\}$  and  $\{q_n\} = \left\{\frac{-1}{3n}\right\}$  in  $\mathfrak{D}$ , all are equal to 0, that is,  $\lim_{n \rightarrow \infty} A p_n = \lim_{n \rightarrow \infty} S p_n = \lim_{n \rightarrow \infty} B q_n = \lim_{n \rightarrow \infty} J q_n = S 0 = J 0 = 0$ .

Clearly, the pairs  $(A, S)$  and  $(B, T)$  satisfy  $JCLR_{ST}$  property.

## 2 | MAIN RESULTS

**Theorem 1.** Let  $A, B, S$  and  $T$  are four self maps on a Menger space  $(\mathfrak{D}, \mathcal{F}, \Delta)$  with  $\Delta(a, b) = \min\{a, b\}$  for all  $a, b \in [0, 1]$  satisfying the following conditions,

- (i)  $A(\mathfrak{D}) \subseteq T(\mathfrak{D}), B(\mathfrak{D}) \subseteq S(\mathfrak{D});$
- (ii) for  $\epsilon > 0$  and for all  $p, q \in \mathfrak{D}$ , there exists a  $\delta \in (0, \epsilon)$  s.t  $\epsilon - \delta < m(p, q, t) \leq \epsilon$  implies  $F(Ap, Bq, t) > \epsilon$ , where  $m(p, q, t) = \min\{F(Sp, Tq, t), F(Ap, Sp, t), F(Bq, Tq, t)\};$
- (iii) one of  $A\mathfrak{D}, B\mathfrak{D}, S\mathfrak{D}$  or  $T\mathfrak{D}$  is a complete subspace of  $\mathfrak{D}$ .

Then  $Av = z = Sv$  and  $Bw = z = Tw$ . Also, if the pair  $(A, S)$  as well as  $(B, T)$  are weakly compatible, then  $Az = Bz = Sz = Tz = z$ , and  $z$  is unique in  $\mathfrak{D}$ .

*Proof.* Since  $A(\mathfrak{D}) \subseteq T(\mathfrak{D})$ . Consider a point  $p_0 \in \mathfrak{D}$ , then there exists  $p_1 \in \mathfrak{D}$  s.t  $A p_0 = T p_1 = q_0$ . For the point  $p_1$ , there exists  $p_2 \in \mathfrak{D}$  such that  $B p_1 = S p_2 = q_1$ . Continuing in this way, we have  $\{\{p_n\}$  and  $q_n\}$  in  $\mathfrak{D}$  s.t

$$q_{2n} = S p_{2n} = B p_{2n-1}; = T p_{2n-1} = A p_{2n-2}$$

We claim that  $\{q_n\}$  is a Cauchy sequence in  $\mathfrak{D}$ .

Let  $F_n = F(q_n, q_{n+1}, t)$  and  $G_n = F(q_n, q_{n+1}, t)$ , where  $t > 0$ .

The two cases arise, suppose that  $F_n = 1$  for some  $n = 2k - 1$ , then  $F(q_{2k-1}, q_{2k}, t) = 1$ . Then  $q_{2k-1} = q_{2k}$  gives  $T p_{2k-1} = A p_{2k-2} = S p_{2k} = B p_{2k-1}$ , so  $T$  and  $B$  have a coincidence point. Again if  $F_n = 1$  for some  $n = 2k$ , then  $F(q_{2k}, q_{2k+1}, t) = 1$ . Then  $q_{2k} = q_{2k+1}$  gives  $T p_{2k+1} = A p_{2k} = S p_{2k} = B p_{2k-1}$ , so  $A$  and  $S$  have a coincidence point.

Next assume that  $F_n \neq 1$  for all  $n$ . If some  $p, q \in \mathfrak{D}, m(p, q, t) = 1$ , then we get  $A p = S p$  and  $T q = B q$ . This proves the result.

If  $m(\rho, q, t) < 1$ , for all  $\rho, q \in \mathfrak{D}$ , then, by,

$$F(\mathcal{A}\rho, \mathcal{B}q, t) > m(\rho, q, t) \quad (1)$$

We have,

$$\begin{aligned} \mathcal{F}_{2n-1} &= F(q_{2n-1}, q_{2n}, t) = F(\mathcal{A}\rho_{2n-2}, \mathcal{B}\rho_{2n-1}, t) \\ &> m(\rho_{2n-2}, \rho_{2n-1}, t) \\ &= \min\{F(\mathcal{S}\rho_{2n-2}, \mathcal{T}\rho_{2n-1}, t), F(\mathcal{A}\rho_{2n-2}, \mathcal{S}\rho_{2n-2}, t), F(\mathcal{B}\rho_{2n-1}, \mathcal{T}\rho_{2n-1}, t)\} \\ &= \min\{F(q_{2n-2}, q_{2n-1}, t), F(q_{2n-1}, q_{2n-2}, t), F(q_{2n}, q_{2n-1}, t)\} \\ &= \min\{\mathcal{F}_{2n-2}, \mathcal{F}_{2n-1}\} = \mathcal{F}_{2n-2}. \end{aligned} \quad (2)$$

So,  $\mathcal{F}_{2n-1} > \mathcal{F}_{2n-2}$

Similarly,  $\mathcal{F}_{2n} > \mathcal{F}_{2n-1}$ .

Therefore one can find that  $\mathcal{F}_n > \mathcal{F}_{n-1}$  for all  $n$ .

Thus in  $[0, 1]$ ,  $\{\mathcal{F}_n\}$  a sequence of positive real numbers is a strictly increasing.

$$\text{Hence } \{\mathcal{F}_n\} \rightarrow \text{some limit say } s. \quad (3)$$

Next we prove that  $s = 1$ . If  $s \neq 1$ , then by (3),  $\exists \delta > 0$  and  $m \in \mathbb{N}$  s.t  $\forall n \geq m$ ,

$$s - \delta < F(q_n, q_{n+1}, t) = \mathcal{F}_n \leq s \quad (4)$$

In particular,  $m(\rho_{2n-2}, \rho_{2n-1}, t) = \min\{\mathcal{F}_{2n}, \mathcal{F}_{2n-1}\} = \mathcal{F}_{2n-1}$ ,

we get  $s - \delta < \mathcal{F}_{2n-1} \leq p$ . Therefore, by using (ii),

$$F(\mathcal{A}\rho_{2n}, \mathcal{B}\rho_{2n-1}, t) = F(q_{2n+1}, q_{2n}, t) = \mathcal{F}_{2n} > s,$$

This is a contradiction. Hence  $s = 1$ , that is,  $\lim_{n \rightarrow \infty} \mathcal{F}_n = \lim_{n \rightarrow \infty} F(q_n, q_{n+1}, t) = 1$ .

Now, for  $k \in \mathbb{Z}^+$ ,

$$F(q_n, q_{n+k}, t) \geq F\left(q_n, q_{n+1}, \frac{t}{k}\right) \Delta F\left(q_{n+1}, q_{n+2}, \frac{t}{k}\right) \Delta \dots \Delta F\left(q_{n+k-1}, q_{n+k}, \frac{t}{k}\right).$$

Since  $\lim_{n \rightarrow \infty} F(q_n, q_{n+1}, t) = 1$  for  $t > 0$ , it follows

$$\lim_{n \rightarrow \infty} F(q_n, q_{n+1}, t) \geq 1 \Delta 1 \Delta \dots \Delta = 1,$$

Then  $\{q_n\}$  is a Cauchy sequence in  $\mathfrak{D}$ .

Now by (iii) assume that  $\mathcal{S}\mathfrak{D}$  is a complete subspace in  $\mathfrak{D}$ , then the subsequence  $q_{2n} = \mathcal{S}\rho_{2n}$  must have a limit  $z$  in  $\mathcal{S}\mathfrak{D}$  and  $v \in \mathcal{S}^{-1}(z)$ , so that  $\mathcal{S}v = z$ . As the sequence  $\{q_{2n}\}$  is contained in  $\{q_n\}$ , and  $\{q_n\}$  is a Cauchy sequence then the sequence  $\{q_n\}$  also converges to  $z$ . First we prove that  $\mathcal{A}v = z$ . If  $\mathcal{A}v \neq z$ . Then, on setting  $\rho = v$  and  $q = \rho_{2n-1}$  in (ii), we have for  $t > 0$ ,

$$\begin{aligned} F(\mathcal{A}v, \mathcal{B}\rho_{2n-1}, t) &> m(v, \rho_{2n-1}, t) \\ &= \min\{F(\mathcal{S}v, \mathcal{T}\rho_{2n-1}, t), F(\mathcal{A}v, \mathcal{S}v, t), F(\mathcal{B}\rho_{2n-1}, \mathcal{T}\rho_{2n-1}, t)\}. \end{aligned}$$

Taking the limit as  $n \rightarrow \infty$ ,

$$F(\mathcal{A}v, z, t) > \min\{F(z, z, t), F(\mathcal{A}v, z, t), F(z, z, t)\} = F(\mathcal{A}v, z, t),$$

this gives a contradiction. Therefore,  $\mathcal{A}v = z = \mathcal{S}v$ .

As  $(\mathfrak{D}) \subseteq \mathcal{T}(\mathfrak{D})$ ,  $\mathcal{A}v = z \Rightarrow z \in \mathcal{J}(\mathfrak{D})$ . Let  $w \in \mathcal{J}^{-1}(z)$ , then  $\mathcal{J}w = z$ .

Next we claim that  $Bw = z$ . If  $Bw \neq z$ , then on setting  $p = q_{2n}$  and  $q = w$  in (ii), we get for  $t > 0$ ,

$$\begin{aligned} F(Aq_{2n}, Bw, t) &= F(q_{2n+1}, Bw, t) > m(q_{2n}, w, t) \\ &= \min\{F(Sq_{2n}, Tw, t), F(Aq_{2n}, Sq_{2n}, t), F(Bw, Tw, t)\}, \end{aligned}$$

Taking the limit as  $n \rightarrow \infty$ ,

$$F(z, Bw, t) > \min\{F(z, z, t), F(z, z, t), F(Bw, z, t)\} = F(Bw, z, t),$$

is a contradiction. Therefore,  $Bw = z = Tw$ .

Hence we have shown that  $z = Sv = Av = Bw = Tw$ .

If we assume  $\mathcal{T}(\mathfrak{D})$  is complete, we get the same. If  $\mathcal{A}(\mathfrak{D})$  is complete, then  $z \in \mathcal{A}(\mathfrak{D}) \subseteq \mathcal{T}(\mathfrak{D})$  and if  $\mathcal{B}(\mathfrak{D})$  is complete, then  $z \in \mathcal{B}(\mathfrak{D}) \subseteq \mathcal{S}(\mathfrak{D})$ . As  $(\mathcal{A}, \mathcal{S})$  and  $(\mathcal{B}, \mathcal{T})$  are weakly compatible, then  $\mathcal{A}z = \mathcal{A}Sv = \mathcal{S}Av = Sz$  and  $\mathcal{B}z = \mathcal{B}Tw = \mathcal{T}Bw = Tz$ .

Finally we claim that  $\mathcal{A}z = z$ . If  $\mathcal{A}z \neq z$ , then on setting  $p = z$  and  $q = w$  in (ii), we have for  $t > 0$ ,

$$\begin{aligned} F(\mathcal{A}z, Bw, t) &= F(\mathcal{A}z, z, t) > m(z, z, t) \\ &= \min\{F(Sz, Tw, t), F(\mathcal{A}z, Sz, t), F(Bw, Tw, t)\} \\ &= \min\{F(\mathcal{A}z, z, t), F(\mathcal{A}z, \mathcal{A}z, t), F(z, z, t)\} = F(\mathcal{A}z, z, t) \end{aligned}$$

this gives a contradiction. Therefore,  $\mathcal{A}z = z$ .

Similarly, we prove  $\mathcal{B}z = z$  and the proof of uniqueness can be found from (ii).

Thus  $\mathcal{A}z = \mathcal{B}z = Sz = Tz = z$ , and  $z$  is unique in  $\mathfrak{D}$ .

**Example 4.** Let  $(\mathfrak{D}, F, \Delta)$  be a Menger space with  $\mathfrak{D} = [2, 20]$  and  $F(p, q, t) = \frac{t}{t+|p-q|}$  for all  $p, q \in \mathfrak{D}, t > 0$  and  $F(p, q, 0) = 0$ , where  $\Delta(a, b) = \min\{a, b\}$  for all  $a, b \in [0, 1]$ . Define  $\mathcal{A}, \mathcal{B}, \mathcal{S}$  and  $\mathcal{T}$  self maps on  $\mathfrak{D}$  by

$$\begin{aligned} \mathcal{A}p &= \begin{cases} 2 & \text{if } p = 2 \text{ or } p > 5 \\ p + 1 & \text{if } 2 < p \leq 5 \end{cases}, & \mathcal{B}p &= \begin{cases} 2 & \text{if } p = 2 \text{ or } p > 5 \\ p + 2 & \text{if } 2 < p \leq 5 \end{cases} \\ \mathcal{S}p &= \begin{cases} 2 & \text{if } p = 2 \\ 8 & \text{if } 2 < p \leq 5 \\ \frac{p+1}{3} & \text{if } p > 5, \end{cases} & \mathcal{T}p &= \begin{cases} 2 & \text{if } p = 2 \text{ or } p > 5 \\ p + 1 & \text{if } 2 < p \leq 5 \end{cases}. \end{aligned}$$

Then  $\mathcal{A}, \mathcal{B}, \mathcal{S}$  and  $\mathcal{T}$  satisfies all the axioms of Theorem 1 and  $\mathcal{A}2 = \mathcal{B}2 = \mathcal{S}2 = \mathcal{T}2 = 2$ , and 2 is unique in  $\mathfrak{D}$ . Also, all are discontinuous at  $p = 2$  and  $\mathcal{S}(\mathfrak{D})$  be complete subspace in  $\mathfrak{D}$ .

Now we are looking to prove Theorem 1 by using common property (E.A), as it relaxes,  $\mathcal{A}(\mathfrak{D}) \subseteq \mathcal{T}(\mathfrak{D})$  or  $\mathcal{B}(\mathfrak{D}) \subseteq \mathcal{S}(\mathfrak{D})$ .

**Theorem 2.** Let  $\mathcal{A}, \mathcal{B}, \mathcal{S}$  and  $\mathcal{T}$  are four self maps on a Menger space  $(\mathfrak{D}, F, \Delta)$  with  $\Delta(a, b) = \min\{a, b\}, \forall a, b \in [0, 1]$  satisfying (ii) and the following conditions,

- (iv) pairs  $(\mathcal{A}, \mathcal{S})$  and  $(\mathcal{B}, \mathcal{T})$  holds (E.A) common property.
- (v)  $\mathcal{S}\mathfrak{D}$  or  $\mathcal{T}\mathfrak{D}$  are closed in  $\mathfrak{D}$ .

Then  $\mathcal{A}u = ru = \mathcal{S}u$  and  $\mathcal{B}v = rv = \mathcal{T}v$ . Also, if the pair  $(\mathcal{A}, \mathcal{S})$  as well as  $(\mathcal{B}, \mathcal{T})$  are weakly compatible, then  $\mathcal{A}r = \mathcal{B}r = \mathcal{S}r = \mathcal{T}r = r$ , and  $r$  is unique in  $\mathfrak{D}$ .

*Proof.* From (iv), there exist two sequences  $\{p_n\}$  and  $\{q_n\}$  in  $\mathfrak{D}$ .t

$$\lim_{n \rightarrow \infty} \mathcal{A}p_n = \lim_{n \rightarrow \infty} \mathcal{S}p_n = \lim_{n \rightarrow \infty} \mathcal{B}q_n = \lim_{n \rightarrow \infty} \mathcal{T}q_n = r \text{ for some } r \in \mathfrak{D}.$$

Now  $\mathcal{S}(\mathfrak{D})$  is closed in  $\mathfrak{D}$ , there exists a point  $u \in \mathfrak{D}$  such that  $r = Su$ . ■

First we claim  $Au = r$ . If  $Au \neq r$ , then on setting  $p = u$  and  $q = q_n$  in (ii), we have for  $t > 0$ ,

$$\begin{aligned} F(Au, Bq_n, t) &> m(u, q_n, t) \\ &= \min\{F(Su, Tq_n, t), F(Au, Su, t), F(Bq_n, Tq_n, t)\} \end{aligned}$$

Taking the limit as  $n \rightarrow \infty$ ,

$$F(Au, r, t) > \min\{F(r, r, t), F(Au, r, t), F(r, r, t)\} = F(Au, r, t),$$

this is a contradiction. Therefore,  $Au = r = Su$ .

Now  $\mathcal{T}(\mathfrak{D})$  is closed in  $\mathfrak{D}$ ,  $\lim_{n \rightarrow \infty} Tq_n = r \in \mathcal{T}(\mathfrak{D})$ , so there exists a point  $v \in \mathfrak{D}$  s.t  $Tv = r = Au = Su$ .

Now we claim that that  $Bv = r$ . If  $Bv \neq r$ , then on setting  $p = u$  and  $q = v$  in (ii), we have for  $t > 0$ ,

$$F(Au, Bv, t) > m(u, v, t) = \min\{F(Su, Tv, t), F(Au, Su, t), F(Bv, Tv, t)\}$$

$$F(r, Bv, t) > \min\{F(r, r, t), F(r, r, t), F(Bv, r, t)\} = F(Bv, r, t)$$

this is a contradiction. Thus,  $Bv = r$  and therefore  $Bv = r = Tv$ .

As  $(A, S)$  and  $(B, T)$  are weakly compatible, then  $Au = Su, Bv = Tv, Ar = ASu = SAu = Sr$  and  $Br = BTv = TBv = Tr$ .

Now we claim that  $Ar = r$ . If  $Ar \neq r$ , then on setting  $p = r$  and  $q = v$  in (ii), we have for  $t > 0$ ,

$$F(Ar, Bv, t) = F(Ar, v, t) > m(r, v, t)$$

$$= \min\{F(Sr, Tv, t), F(Ar, Sr, t), F(Bv, Tv, t)\}$$

$$= \min\{F(Ar, r, t), F(Ar, Ar, t), F(r, r, t)\} = F(Ar, r, t),$$

gives a contradiction. Therefore,  $Ar = r$ .

Similarly, one can prove  $Br = r$  and the uniqueness can be taken out from (ii).

Thus  $Ar = Br = Sr = Tr = r$ , and  $r$  is unique in  $\mathfrak{D}$ .

**Example 5.** Let  $(\mathfrak{D}, F, \Delta)$  be a Menger space with  $\mathfrak{D} = [2, 20]$  and  $F(p, q, t) = \frac{t}{t+|p-q|}$  for all  $p, q \in \mathfrak{D}, t > 0$  and  $F(p, q, 0) = 0$  where  $\Delta(a, b) = \min\{a, b\}$  for all  $a, b \in [0, 1]$ . Define  $A, B, S$  and  $T$  self maps on  $\mathfrak{D}$  by

$$Ap = \begin{cases} 2 & \text{if } p = 2 \text{ or } p > 5 \\ p + 1 & \text{if } 2 < p \leq 5 \end{cases}, \quad Bp = \begin{cases} 2 & \text{if } p = 2 \text{ or } p > 5 \\ p + 2 & \text{if } 2 < p \leq 5 \end{cases}$$

$$Sp = \begin{cases} 2 & \text{if } p = 2 \text{ or } p > 5 \\ p + 1 & \text{if } 2 < p \leq 5 \end{cases}, \quad Tp = \begin{cases} 2 & \text{if } p = 2 \text{ or } p > 5 \\ 9 & \text{if } 2 < p \leq 5 \end{cases}.$$

Take  $\left\{p_n = 5 + \frac{1}{n}\right\}$  and  $\left\{q_n = 5 + \frac{1}{n}\right\}$ . Then  $\lim_{n \rightarrow \infty} Ap_n = \lim_{n \rightarrow \infty} Sp_n = \lim_{n \rightarrow \infty} Bq_n = \lim_{n \rightarrow \infty} Tq_n = 2 \in \mathfrak{D}$ . Then  $A, B, S$  and  $T$  satisfy all the axioms of Theorem 2 and  $A2 = B2 = S2 = T2 = 2$ , and 2 is unique in  $\mathfrak{D}$ . All are discontinuous at  $p = 2$  and  $S(\mathfrak{D})$  be complete subspace in  $\mathfrak{D}$ . Here  $S\mathfrak{D}$  and  $T\mathfrak{D}$  are closed in  $\mathfrak{D}$ . Also,  $B(\mathfrak{D}) \not\subseteq S(\mathfrak{D})$  or  $A(\mathfrak{D}) \not\subseteq T(\mathfrak{D})$ .

Next, an effort was made to eliminate the closeness of the subspaces from Theorem 2 by using the  $JCLR_{ST}$  property.

**Theorem 3.** Let  $A, B, S$  and  $T$  are four self maps on a Menger space  $(\mathfrak{D}, F, \Delta)$  with  $\Delta(a, b) = \min\{a, b\}, \forall a, b \in [0, 1]$  satisfying (ii) and the property,

(vi) pairs  $(A, S)$  and  $(B, T)$  holds  $JCLR_{ST}$  property.

Then  $Ax = Sx$  and  $Bx = Tx$ . Also, if the pair  $(A, S)$  as well as  $(B, T)$  are weakly compatible, then  $Au = Bu = Su = Tu = u$ , and  $u$  is unique in  $\mathfrak{D}$ .

*Proof.* As  $(A, S)$  and  $(B, T)$  holds  $JCLR_{ST}$  property, there exist two sequences  $\{\rho_n\}$  and  $\{q_n\}$  in  $\mathfrak{D}$  s.t

$$\lim_{n \rightarrow \infty} F(A\rho_n, u, t) = \lim_{n \rightarrow \infty} F(S\rho_n, u, t) = \lim_{n \rightarrow \infty} F(Bq_n, u, t) = \lim_{n \rightarrow \infty} F(Tq_n, u, t) = 1,$$

where  $u = Sx = Tx$ , for some  $x \in \mathfrak{D}$ . ■

First we claim that  $Ax = Sx$ . If  $Ax \neq Sx$ , then on setting  $\rho = x$  and  $q = q_n$  in (ii), we have for  $t > 0$ ,

$$\begin{aligned} F(Ax, Bq_n, t) &> m(x, q_n, t) \\ &= \min\{F(Sx, Tq_n, t), F(Ax, Sx, t), F(Bq_n, Tq_n, t)\}. \end{aligned}$$

Taking the limit as  $n \rightarrow \infty$ ,

$$F(Ax, Sx, t) > \min\{F(Sx, Sx, t), F(Ax, Sx, t), F(Sx, Sx, t)\} = F(Ax, Sx, t)$$

this is a contradiction. Therefore,  $Ax = Sx$ .

Now we claim that that  $Bx = Tx$ . If  $Bx \neq Tx$ , then on setting  $\rho = x$  and  $q = x$  in (ii), we have for  $t > 0$ ,

$$\begin{aligned} F(Ax, Bx, t) &> m(x, x, t) = \min\{F(Sx, Tx, t), F(Ax, Sx, t), F(Bx, Tx, t)\} \\ F(Tx, Bx, t) &> \min\{F(Sx, Sx, t), F(Tx, Tx, t), F(Bx, Tx, t)\} = F(Bx, Tx, t), \end{aligned}$$

is a contradiction. Therefore,  $Bx = Tx$ .

Now  $u = Sx = Tx = Bx = Tx$ . As  $(A, S)$  and  $(B, T)$  are weakly compatible, then  $Ax = Sx, Bx = Tx, Au = ASx = SAx = Su$  and  $Bu = BTx = TBx = Tu$ .

Now we claim that  $Au = u$ . If  $Au \neq u$ , then on setting  $\rho = u$  and  $q = x$  in (ii), we have for  $t > 0$ ,

$$\begin{aligned} F(Au, Bx, t) &= F(Au, x, t) > m(u, x, t) \\ &= \min\{F(Su, Jx, t), F(Au, Su, t), F(Bx, Jx, t)\} \\ &= \min\{F(Au, u, t), F(Au, Au, t), F(u, u, t)\} = F(Au, u, t) \end{aligned}$$

is a contradiction. Therefore,  $Au = u$ .

Similarly, one can prove  $Bu = u$  and the uniqueness can be proved from (ii).

Thus  $Au = Bu = Su = Tu = u$ , and  $u$  is unique in  $\mathfrak{D}$ .

**Example 6.** Let  $(\mathfrak{D}, F, \Delta)$  be a Menger space with  $\mathfrak{D} = [2, 20]$  and  $F(\rho, q, t) = \frac{t}{t+|\rho-q|}$  for all  $\rho, q \in \mathfrak{D}, t > 0$  and  $F(\rho, q, 0) = 0$ , where  $\Delta(a, b) = \min\{a, b\}$  for all  $a, b \in [0, 1]$ . Define  $A, B, S$  and  $T$  self maps on  $\mathfrak{D}$  by

$$\begin{aligned} A\rho &= \begin{cases} 2 & \text{if } \rho = 2 \text{ or } \rho > 5 \\ \rho + 1 & \text{if } 2 < \rho \leq 5 \end{cases}, & B\rho &= \begin{cases} 2 & \text{if } \rho = 2 \text{ or } \rho > 5 \\ \rho + 2 & \text{if } 2 < \rho \leq 5 \end{cases} \\ S\rho &= \begin{cases} 2 & \text{if } \rho = 2 \text{ or } \rho > 5 \\ \rho + 1 & \text{if } 2 < \rho \leq 5 \end{cases} & T\rho &= \begin{cases} 2 & \text{if } \rho = 2 \text{ or } \rho > 5 \\ \rho + 9 & \text{if } 2 < \rho \leq 5 \end{cases}. \end{aligned}$$

Take  $\{\rho_n = 5 + \frac{1}{n}\}$  and  $\{q_n = 5 + \frac{1}{n}\}$ . Then  $\lim_{n \rightarrow \infty} A\rho_n = \lim_{n \rightarrow \infty} S\rho_n = \lim_{n \rightarrow \infty} Bq_n = \lim_{n \rightarrow \infty} Tq_n = 2 \in \mathfrak{D}$ . Thus  $A, B, S$  and  $T$  satisfy all axioms of Theorem 3 and  $A2 = B2 = S2 = T2 = 2$ , and 2 is unique in  $\mathfrak{D}$ . Also, all are discontinuous at  $\rho = 2$  and  $S(\mathfrak{D})$  be complete subspace in  $\mathfrak{D}$ . Here  $S\mathfrak{D}$  and  $T\mathfrak{D}$  are closed in  $\mathfrak{D}$ . Also,  $B(\mathfrak{D}) \not\subseteq S(\mathfrak{D})$  or  $A(\mathfrak{D}) \not\subseteq T(\mathfrak{D})$ .

### 3 | CONCLUSION

We have proved some common fixed point theorems for self-maps in Menger spaces with minimum  $t$ -norm satisfying some Meir-Keeler type contractive condition in which two pairs of mappings are weakly compatible and have coincidence point. We have also proved the results with E.A property and  $JCLR_{ST}$  property.

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#### CONFLICT OF INTEREST

All the authors declare that they have no competing interests regarding this manuscript.

#### AUTHORS CONTRIBUTIONS

All authors contributed equally to the writing of this manuscript. All authors read and approved the final version

#### ORCID

Vishal Gupta  <https://orcid.org/0000-0001-9727-2827>

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## AUTHOR BIOGRAPHIES



**Vishal Gupta**, having more than 11 years of teaching experience, is currently working as professor in Department of Mathematics, Maharishi Markandeshwar (Deemed to be University), Mullana, India. He received his PhD degree in Mathematics in 2010. Also, he earned the degree of MPhil in Mathematics and MEd. He has published one research book with international publisher and his immense contribution in journals of national and international repute is more than 70. He has presented more than 50 research papers in national and international conferences. His research interests are fixed point theory, operator theory, aggregation function, fuzzy set theory and fuzzy mappings, topology, applications of fixed point theory in medical science, in graph theory and differential and integral equations.



**Mohammad S. Khan**, obtained MSc degree from Indian Institute of Technology, Kanpur in 1970 and then from 1971 to 1978 he has worked as a Lecturer at Aligarh Muslim University, India. In 1979, La Trobe University, Melbourne, Australia awarded him a PhD scholarship to work under the able guidance of well-known mathematician Professor Sidney Allen Morris. In 1985, he joined King Abdul Aziz University, Saudi Arabia and stayed there until 1990. From August 1990 until today, he has worked as Professor, Department of Mathematics Statistics, Sultan Qaboos University, P. O. Box 36, Al-Khoud 123, Muscat Sultanate of Oman (Oman). His research interests are fixed point theory, operator theory, functional Analysis and fuzzy mathematics and his enormous contribution in journals of national and international repute is more than 225.



**Balbir Singh**, presently working as an Associate Professor in the School of Physical Sciences (Department of Mathematics), Starex University, Gurugram Haryana. He has published 21 National and international papers in Scopus and UGC listed journals and three books on Engineering Mathematics and has 12 years teaching experience to teach the PG and UG classes. Dr Balbir Singh has completed his PhD degree from Maharishi Markandeshwar University, Mullana (Haryana), under the joint supervision of Dr. Sanjay Kumar and Dr. Vishal Gupta.



**Sanjay Kumar**, working as an Professor in the Department of Mathematics, Deenbandu Chhotu Ram University, Murthal, Sonapat Haryana (INDIA). He had also worked in NCERT from 2004 to 2009 as an Assistant Professor in Central Institute of Educational Technology and Department of Educational Mathematics and Sciences. He is members of Mathematics development committee of NCERT, Mathematics Text Books and Exemplar books. He has published more than 160 research papers in various national international journal of repute. He has guided 10 PhD students in area of Fixed Point Theory.

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