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COMPATIBLE MAPPINGS AND ITS VARIANTS FOR GENERALIZED $\psi -\emptyset -$WEAK CONTRACTION CONDITION

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Abstract. In the present paper, first we introduce generalized $\psi - \emptyset -$weak contraction condition that involves cubic and quadratic terms of distance function $d(x,y)$ and then proved common fixed point theorems for compatible mappings. Secondly, we deal with variants of compatible mappings type (K), type (R) and type (E). At the end, we provide applications of our results.

Keywords: $\psi -\ \emptyset -$weak contraction; compatible mappings.

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1. INTRODUCTION

Banach fixed point theorem is the basic tool to study fixed point theory and shows the existence and uniqueness of a fixed point under appropriate conditions. This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering. Most of the problems of applied mathematics reduce to inequality which in turn their solutions

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give rise to the fixed points of certain mappings. It was the new era of the fixed point theory literature when the notion of commutativity mappings was used by Jungck [5] to obtain a generalization of Banach's fixed point theorem for a pair of mappings. The first ever attempt to relax the commutativity to weak commutativity was initiated by Sessa [21]. In 1986, Jungck [6] introduced more generalized commutativity, so called compatibility. Ever since the introduction of compatibility, the study of common fixed points has developed around compatible maps and its weaker forms and it has become an area of vigorous research activity. Notice that the notions of weak commutativity and compatibility differ in one respect. Weak commutativity is essentially a point property, while the notion of compatibility uses the machinery of sequences. However, fixed point theory for non compatible mappings is equally interesting and Pant [18] has initiated some work along these lines. It may be observed that the mappings f and g are said to be non compatible if there exists a sequence $\{x_n\}$ in $\mathfrak{V}$ such that for some t in $\mathfrak{V}$, but $\lim\limits_{n\to\infty} d(fgx_n, gfx_n)$ is either non-zero or nonexistent. In 1996, Jungck [8] introduced the notion of weakly compatible mappings and showed that compatible maps are weakly compatible, but not converse may not be true. In 1998, Pant [17] introduced a new notion of continuity and called it reciprocally continuous mappings. In 2001, Sahu et al. [22] introduced the notion of intimate mappings in metric spaces. Intimate mappings are more improved version of weakly commuting, semi-compatibility and R-commutativity etc. Sahu et al. [22] have also shown that intimate mappings are more general than compatible mappings. The most crucial feature of intimate mappings is that these mappings do not necessarily commute at a coincidence point. It is the generalization of compatible mappings of type (A). In 2004, Rohan et al. [20] introduced the concept of compatible mappings of type (R) by using the notion of compatible mappings and compatible mappings of type (P) together. In 2007, Singh and Singh [23] introduced the concept of compatible mappings of type (E) by rearranging terms of compatible mappings of type (P) and compatible mappings. In 2014, Jha et al. [11] introduced the concept of compatible mappings of type (K) by modification in compatible mappings of type (P) in a metric space. In 1993, Jungck et al. [10] introduced the notion of compatible mappings of type(A) which is equivalent to concept of compatible mappings under some conditions.

Banach fixed point theorem states that every contraction mapping on a complete metric space has a unique fixed point. Let $(\mathfrak{V}, d)$ be a complete metric space. If $\mathcal{T}: \mathfrak{V} \to \mathfrak{V}$ satisfies $d(\mathcal{T}(x), \mathcal{T}(y)) \leq k(d(x, y))$ for all $x, y \in \mathfrak{V}$, $0 \leq k < 1$, then it has a unique fixed point.

In 1969, Boyd and Wong [3] replaced the constant k in Banach contraction principle by a implicit function ψ as follows:

Let $(\mathfrak{V}, d)$ be a complete metric space and $\psi : [0, \infty) \to [0, \infty)$ be upper semi continuous from the right such that $0 \leq \psi(t) < t$ for all $t > 0$.

If $\mathcal{T}: \mathfrak{V} \to \mathfrak{V}$ satisfies $d\big(\mathcal{T}(x), \mathcal{T}(y)\big) \leq \psi\big(d(x, y)\big)$, for all $x, y \in \mathfrak{V}$, then it has a unique fixed point.

In 1997, Alber and Gueree-Delabriere [1] introduced the concept of weak contraction as follows: A map $\mathcal{T}: \mathfrak{V} \to \mathfrak{V}$ is said to be weak contraction if for each $x, y \in \mathfrak{V}$, there exists a function $\emptyset : [0, \infty) \to [0, \infty)$, $\emptyset(t) > 0$ for all $t > 0$ and $\emptyset(0) = 0$ such that

$d(\mathcal{T}(x), \mathcal{T}(y)) \leq d(x, y) - \emptyset(d(x, y))$.

In connection with control function $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ different authors have considered some of the following properties:

(i) ψ is non decreasing

(ii) $\psi(t) < 0$,for all $t > 0$.

(iii) $\psi(0) = 0$

(iv) ψ is continuous

(v) $\lim_{n\to\infty} \psi^n(t) = 0$,for all $t \geq 0$.

(vi) $\sum_{n=0}^{\infty} \psi^n(t)$ converges for all $t > 0$,$\psi^n$is the nth iterate

(vii) $\psi(t) = 0$ iff $t = 0$

(viii) $\psi(t) > 0$ for all $t \in \mathbb{R}^+\backslash\{0\}$

(ix) $\lim_{r\to t^+} \psi(t) < 0$,for all $t > 0$.

(x) $\lim_{t\to\infty} \psi(t) = \infty$.

(xi) ψ is lower semi continuous

Here we note that

(i) and (ii) implies (iii) ;

(ii) and (iv) implies (iii)

(i) and (v) implies (ii)

A function ψ satisfying (i) and (v) that is ψ is non decreasing and $\lim_{n\to\infty} \psi^n(t) = 0$, for all $t \geq 0$ is called as a comparison function.

2. PRELIMINARIES

In 1996, Jungck [8] introduced the notion of weakly compatible mappings and showed that compatible maps are weakly compatible, but converse may not be true.

Definition 2.1[8] Two self-mappings f and g of a metric space $(\mathfrak{V}, d)$ are called weakly compatible if they commute at their coincidence point i.e., if $fu = gu$ for some $u \in \mathfrak{V}$, then $fgu = gfu$.

In 1982, S. Sessa [21] generalized the concept of commutativity to the notion of weak commutativity of maps. Thereafter, in 1986, Jungck [6] generalized and extend the notion of weak commutativity to compatible mappings.

Definition 2.2[6] Two self-mappings f and g of a metric space $(\mathfrak{V}, d)$ are called compatible if $lim_n d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in $\mathfrak{V}$ such that

$lim_n fx_n = lim_n gx_n = t$,for some t in $\mathfrak{V}$.

Now we state some properties for compatible mappings that are fruitful for further study.

Proposition 2.1[6] Let $\mathcal{S}$ and $\mathcal{T}$ be compatible mappings of a metric space $(\mathfrak{V}, d)$ into itself. If $\mathcal{S}t = \mathcal{T}t$ for some t in $\mathfrak{V}$, then $\mathcal{S}\mathcal{T}t = \mathcal{S}\mathcal{S}t = \mathcal{T}\mathcal{T}t = \mathcal{T}\mathcal{S}t$.

Proposition 2.2 [6] Let $\mathcal{S}$ and $\mathcal{T}$ be compatible mappings of a metric space $(\mathfrak{V}, d)$ into itself. Suppose that $lim_n \mathcal{S}x_n = lim_n \mathcal{T}x_n = t$ for some t in $\mathfrak{V}$. Then the following holds:

(i) $lim_n \mathcal{T}\,\mathcal{S}x_n = \mathcal{S}t$ if S is continuous at t;

(ii) $lim_n \mathcal{S}Tx_n = \mathcal{T}t$ if $\mathcal{T}$ is continuous at t;

(iii) $\mathcal{S}\mathcal{T}t = \mathcal{T}\mathcal{S}t$ and $\mathcal{S}t = \mathcal{T}t$ if $\mathcal{S}$ and $\mathcal{T}$ are continuous at t.

Now we introduce the generalized $\psi - \emptyset$-weak contraction for a pairs of mappings in the following way:

Let $\mathcal{S}, \mathcal{T}, \mathcal{A}$ and $\mathcal{B}$ are four self mappings on a metric space $(\mathfrak{V}, d)$ satisfying the following conditions:

(C1) $\mathcal{S}(\mathfrak{V}) \subset \mathcal{B}(\mathfrak{V}), \mathcal{T}(\mathfrak{V}) \subset \mathcal{A}(\mathfrak{V})$;

(C2) $$d^3(\mathcal{S}u, \mathcal{T}v) \leq \psi \left\{ \begin{array}{c} d^2(\mathcal{A}u, \mathcal{S}u)d(\mathcal{B}v, \mathcal{T}v), \\ d(\mathcal{A}u, \mathcal{S}u)d^2(\mathcal{B}v, \mathcal{T}v), \\ d(\mathcal{A}u, \mathcal{S}u)d(\mathcal{A}u, \mathcal{T}v)d(\mathcal{B}v, \mathcal{S}u), \\ d(\mathcal{A}u, \mathcal{T}v)d(\mathcal{B}v, \mathcal{S}u)d(\mathcal{B}v, \mathcal{T}v) \end{array} \right\} - \emptyset\,\{m(\mathcal{A}u, \mathcal{B}v)\},$$

where $$m(\mathcal{A}u, \mathcal{B}v) = max \left\{ \begin{array}{c} d^2(\mathcal{A}u, \mathcal{B}v), d(\mathcal{A}u, \mathcal{S}u)d(\mathcal{B}v, \mathcal{T}v), \\ d(\mathcal{A}u, \mathcal{T}v)d(\mathcal{B}v, \mathcal{S}u), \\ \frac{1}{2}\left[\begin{array}{l} d(\mathcal{A}u, \mathcal{S}u)d(\mathcal{A}u, \mathcal{T}v) \\ +d(\mathcal{B}v, \mathcal{S}u)d(\mathcal{B}v, \mathcal{T}v) \end{array} \right] \end{array} \right\}$$

for all $u, v \in \mathfrak{V}$, where ψ: $[0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function with $\psi(t) < t$ for each $t > 0$ and $\emptyset$: $[0, \infty) \to [0, \infty)$ is a continuous function with $\emptyset(t) = 0 \Leftrightarrow t = 0$ and $\emptyset(t) > 0$ for each $t > 0$.

In this section, we prove a result for compatible mappings that satisfy generalized $\psi - \emptyset$ −weak contraction involving cubic and quadratic terms of distance function.

Theorem 2.1 Let $(\mathfrak{V}, d)$ be a complete metric space. Let $\mathcal{S}, \mathcal{T}, \mathcal{A}$ and $\mathcal{B}$ are four mappings of a complete metric space $(\mathfrak{V}, d)$ into itself satisfying (C1) and (C2) and the following conditions:

(2.1) one of $\mathcal{S}, \mathcal{T}, \mathcal{A}$ and $\mathcal{B}$ is continuous.

Assume that the pairs $(\mathcal{A},\mathcal{S})$ and $(\mathcal{B},\mathcal{T})$ are compatible. Then $\mathcal{S},\mathcal{T},\mathcal{A}$ and $\mathcal{B}$ have a unique common fixed point in $\mathfrak{V}$.

Proof. Let $x_0 \in \mathfrak{V}$ be an arbitrary point. From (C1) we can find x_1 such that $\mathcal{S}(x_0) = \mathcal{B}(x_1) = y_0$ for this x_1 one can find $x_2 \in \mathfrak{V}$ such that $\mathcal{T}(x_1) = \mathcal{A}(x_2) = y_1$.Continuing in this way, one can construct a sequence $\{x_n\}$ such that

$$y_{2n} = \mathcal{S}(x_{2n}) = \mathcal{B}(x_{2n+1}),$$

$$y_{2n+1} = \mathcal{T}(x_{2n+1}) = \mathcal{A}(x_{2n+2}),\ \text{for each } n \geq 0. \tag{2.2}$$

For brevity, we write $\alpha_{2n} = d(y_{2n}, y_{2n+1})$

First, we prove that $\{\alpha_{2n}\}$ is non-increasing sequence and converges to zero.

Case I If n is even, taking $u = x_{2n}$ and $v = x_{2n+1}$ in (C2), we get

$$d^3(\mathcal{S}x_{2n},\mathcal{T}x_{2n+1}) \leq \psi\left\{\begin{matrix} d^2(\mathcal{A}x_{2n},\mathcal{S}x_{2n})d(\mathcal{B}x_{2n+1},\mathcal{T}x_{2n+1}) \\ ,d(\mathcal{A}x_{2n},\mathcal{S}x_{2n})d^2(\mathcal{B}x_{2n+1},\mathcal{T}x_{2n+1}), \\ d(\mathcal{A}x_{2n},\mathcal{S}x_{2n})d(\mathcal{A}x_{2n},\mathcal{T}x_{2n+1})d(\mathcal{B}x_{2n+1},\mathcal{S}x_{2n}), \\ d(\mathcal{A}x_{2n},\mathcal{T}x_{2n+1})d(\mathcal{B}x_{2n+1},\mathcal{S}x_{2n})d(\mathcal{B}x_{2n+1},\mathcal{T}x_{2n+1}) \end{matrix}\right\}$$
$$-\emptyset\{m(\mathcal{A}x_{2n},\mathcal{B}x_{2n+1})\},$$

where

$$m(\mathcal{A}x_{2n},\mathcal{B}x_{2n+1}) = max\left\{\begin{matrix} d^2(\mathcal{A}x_{2n},\mathcal{B}x_{2n+1}),d(\mathcal{A}x_{2n},\mathcal{S}x_{2n})d(\mathcal{B}x_{2n+1},\mathcal{T}x_{2n+1}), \\ d(\mathcal{A}x_{2n},\mathcal{T}x_{2n+1})d(\mathcal{B}x_{2n+1},\mathcal{S}x_{2n}), \\ \frac{1}{2}\begin{bmatrix} d(\mathcal{A}x_{2n},\mathcal{S}x_{2n})d(\mathcal{A}x_{2n},\mathcal{T}x_{2n+1}) \\ +d(\mathcal{B}x_{2n+1},\mathcal{S}x_{2n})d(\mathcal{B}x_{2n+1},\mathcal{T}x_{2n+1}) \end{bmatrix} \end{matrix}\right\}$$

Using (2.2), we have

$$d^3(y_{2n},y_{2n+1}) \leq \psi\left\{\begin{matrix} d^2(y_{2n-1},y_{2n})d(y_{2n},y_{2n+1}) \\ ,d(y_{2n-1},y_{2n})d^2(y_{2n},y_{2n+1}), \\ d(y_{2n-1},y_{2n})d(y_{2n-1},y_{2n+1})d(y_{2n},y_{2n}), \\ d(y_{2n-1},y_{2n+1})d(y_{2n},y_{2n})d(y_{2n},y_{2n+1}) \end{matrix}\right\}$$
$$-\emptyset\{m(y_{2n-1},y_{2n})\},$$

where

$$m(y_{2n-1},y_{2n}) = max\left\{\begin{matrix} d^2(y_{2n-1},y_{2n}),d(y_{2n-1},y_{2n})d(y_{2n},y_{2n+1}), \\ d(y_{2n-1},y_{2n+1})d(y_{2n},y_{2n}), \\ \frac{1}{2}\begin{bmatrix} d(y_{2n-1},y_{2n})d(y_{2n-1},y_{2n+1}) \\ +d(y_{2n},y_{2n})d(y_{2n},y_{2n+1}) \end{bmatrix} \end{matrix}\right\}.$$

On using $\alpha_{2n} = d(y_{2n}, y_{2n+1})$ in the above inequality we have

$$\alpha_{2n}^3 \le \psi\{\alpha_{2n-1}^2\alpha_{2n}, \alpha_{2n-1}\alpha_{2n}^2, 0,0\} - \emptyset\{m(y_{2n-1}, y_{2n})\}, \tag{2.3}$$

where $m(y_{2n-1}, y_{2n}) = max\left\{\alpha_{2n-1}^2, \alpha_{2n-1}\alpha_{2n}, 0, \frac{1}{2}[\alpha_{2n-1}d(y_{2n-1}, y_{2n+1}) + 0]\right\}$.

By using triangular inequality and property ψ *and* $\emptyset$, we get

$$d(y_{2n-1}, y_{2n+1}) \le d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})$$

$$= \alpha_{2n-1} + \alpha_{2n} \quad \text{and}$$

$$m(y_{2n-1}, y_{2n}) \le m'(x, y) = max\left\{\alpha_{2n-1}^2, \alpha_{2n-1}\alpha_{2n}, 0, \frac{1}{2}[\alpha_{2n-1}(\alpha_{2n-1} + \alpha_{2n}), 0]\right\}.$$

If $\alpha_{2n-1} < \alpha_{2n}$ and using property of ψ and $\emptyset$, then (2.3) reduces to

$\alpha_{2n}^3 < \alpha_{2n}^3$, a contradiction, therefore, $\alpha_{2n} \le \alpha_{2n-1}$.

In a similar way, if n is odd, then we can obtain$\alpha_{2n+1} < \alpha_{2n}$.

It follows that the sequence $\{\alpha_{2n}\}$ is decreasing.

Let $\lim_{n\to\infty} \alpha_{2n} = r$, for some $r \ge 0$.

Suppose $r > 0$; then from inequality (C2) and (2.2) and (2.3) we have

$r^3 \le \psi\,(r^3) - \emptyset(r^2) < r^3$, a contradiction, therefore we get $r = 0$. Therfore

$$\lim_{n\to\infty} \alpha_{2n} = \lim_{n\to\infty} d(y_{2n}, y_{2n+1}) = r = 0. \tag{2.4}$$

Now we show that $\{y_n\}$ is a Cauchy sequence. Suppose that $\{y_n\}$ is not a Cauchy sequence. For given $\epsilon > 0$, we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k , $n(k) > m(k) > k$.

$$d(y_{m(k)}, y_{n(k)}) \ge \epsilon, \quad d\big(y_{m(k)}, y_{n(k)-1}\big) < \epsilon \tag{2.5}$$

Now $\quad \epsilon \le d\big(y_{m(k)}, y_{n(k)}\big) \le d\big(y_{m(k)}, y_{n(k)-1}\big) + d(y_{n(k)-1}, y_{n(k)})$

Letting $k \to \infty$, and using (2.4) and (2.5), we get $\lim_{k\to\infty} d\big(y_{m(k)}, y_{n(k)}\big) = \epsilon$

Now from the triangular inequality, we have,

$$\left|d\big(y_{n(k)}, y_{m(k)+1}\big) - d\big(y_{m(k)}, y_{n(k)}\big)\right| \le d\big(y_{m(k)}, y_{m(k)+1}\big).$$

Taking limits as $k \to \infty$ and using (2.4) and (2.5), we have

$\lim_{k\to\infty} d(\mathcal{y}_{n(k)}, \mathcal{y}_{m(k)+1}) = \epsilon.$

On using triangular inequality, we have

$$\left|d(\mathcal{y}_{m(k)}, \mathcal{y}_{n(k)+1}) - d(\mathcal{y}_{m(k)}, \mathcal{y}_{n(k)})\right| \leq d(\mathcal{y}_{n(k)}, \mathcal{y}_{n(k)+1}).$$

Proceeding limits as $k \to \infty$ and using (2.4) and (2.5), we get

$\lim_{k\to\infty} d(\mathcal{y}_{m(k)}, \mathcal{y}_{n(k)+1}) = \epsilon.$

Similarly, we have

$$\left|d(\mathcal{y}_{m(k)+1}, \mathcal{y}_{n(k)+1}) - d(\mathcal{y}_{m(k)}, \mathcal{y}_{n(k)})\right| \leq d(\mathcal{y}_{m(k)}, \mathcal{y}_{m(k)+1}) + d(\mathcal{y}_{n(k)}, \mathcal{y}_{n(k)+1}).$$

Taking limit as $k \to \infty$ in the above inequality and using (2.4) and (2.5), we have

$$\lim_{k\to\infty} d(\mathcal{y}_{n(k)+1}, \mathcal{y}_{m(k)+1}) = \epsilon.$$

On putting $u = x_{m(k)}$ and $v = x_{n(k)}$ in (C2), we get

$$d^3(\mathcal{S}x_{m(k)}, \mathcal{T}x_{n(k)}) \leq \psi\left\{\begin{array}{c} d^2(\mathcal{A}x_{m(k)}, \mathcal{S}x_{m(k)})d(\mathcal{B}x_{n(k)}, \mathcal{T}x_{n(k)}) \\ , d(\mathcal{A}x_{m(k)}, \mathcal{S}x_{m(k)})d^2(\mathcal{B}x_{n(k)}, \mathcal{T}x_{n(k)}), \\ d(\mathcal{A}x_{m(k)}, \mathcal{S}x_{m(k)})d(\mathcal{A}x_{m(k)}, \mathcal{T}x_{n(k)})d(\mathcal{B}x_{2n+1}, \mathcal{S}x_{m(k)}), \\ d(\mathcal{A}x_{m(k)}, \mathcal{T}x_{n(k)})d(\mathcal{B}x_{n(k)}, \mathcal{S}x_{m(k)})d(\mathcal{B}x_{n(k)}, \mathcal{T}x_{n(k)}) \end{array}\right\}$$

$$-\emptyset\{m(\mathcal{A}x_{m(k)}, \mathcal{B}x_{n(k)})\},$$

where

$$m(\mathcal{A}x_{m(k)}, \mathcal{B}x_{n(k)}) = max\left\{\begin{array}{c} d^2(\mathcal{A}x_{m(k)}, \mathcal{B}x_{n(k)}), \\ d(\mathcal{A}x_{m(k)}, \mathcal{S}x_{2n})d(\mathcal{B}x_{n(k)}, \mathcal{T}x_{n(k)}), \\ d(\mathcal{A}x_{m(k)}, \mathcal{T}x_{n(k)})d(\mathcal{B}x_{n(k)}, \mathcal{S}x_{m(k)}), \\ \frac{1}{2}\left[\begin{array}{c} d(\mathcal{A}x_{m(k)}, \mathcal{S}x_{m(k)})d(\mathcal{A}x_{m(k)}, \mathcal{T}x_{n(k)}) \\ +d(\mathcal{B}x_{n(k)}, \mathcal{S}x_{m(k)})d(\mathcal{B}x_{n(k)}, \mathcal{T}x_{n(k)}) \end{array}\right] \end{array}\right\}$$

Using (2.2), we obtain

$$d^3(\mathcal{y}_{m(k)}, \mathcal{y}_{n(k)}) \leq \psi\left\{\begin{array}{c} d^2(\mathcal{y}_{m(k)-1}, \mathcal{y}_{m(k)})d(\mathcal{y}_{n(k)-1}, \mathcal{y}_{n(k)}), \\ d(\mathcal{y}_{m(k)-1}, \mathcal{y}_{m(k)})d^2(\mathcal{y}_{n(k)-1}, \mathcal{y}_{n(k)}) \\ d(\mathcal{y}_{m(k)-1}, \mathcal{y}_{m(k)})d(\mathcal{y}_{m(k)-1}, \mathcal{y}_{n(k)})d(\mathcal{y}_{n(k)-1}, \mathcal{y}_{m(k)}), \\ d(\mathcal{y}_{m(k)-1}, \mathcal{y}_{n(k)})d(\mathcal{y}_{n(k)-1}, \mathcal{y}_{m(k)})d(\mathcal{y}_{n(k)-1}, \mathcal{y}_{n(k)}) \end{array}\right\}$$

$$-\emptyset\{m(\mathcal{A}x_{m(k)}, \mathcal{B}x_{n(k)})\}$$

where,

$$m(\mathcal{A}x_{m(k)}, \mathcal{B}x_{n(k)}) = max \left\{ \begin{array}{c} d^2(y_{m(k)-1}, y_{n(k)-1}), \\ d(y_{m(k)-1}, y_{m(k)}) d(y_{n(k)-1}, y_{n(k)}), \\ d(y_{m(k)-1}, y_{n(k)}) d(y_{n(k)-1}, y_{m(k)}), \\ \frac{1}{2}\left[\begin{array}{l} d(y_{m(k)-1}, y_{m(k)}) d(y_{m(k)-1}, y_{n(k)}) \\ +d(y_{n(k)-1}, y_{m(k)}) d(y_{n(k)-1}, y_{n(k)}) \end{array} \right] \end{array} \right\}$$

Letting $k \to \infty$, and using property of ψ and $\emptyset$, we have

$$\epsilon^3 \leq 0 - \emptyset(\epsilon^2)$$

$$= -\emptyset(\epsilon^2), \text{ which is a contradiction.}$$

Hence the sequence $\{y_n\}$ is a Cauchy sequence in $\mathfrak{V}$, but $(\mathfrak{V}, d)$ is a complete metric space, therefore, $\{y_n\}$ converges to a point z in $\mathfrak{V}$ as $n \to \infty$. Consequently, the subsequences $\{\mathcal{S}x_{2n}\}, \{\mathcal{A}x_{2n}\}, \{\mathcal{T}x_{2n+1}\}$ and $\{\mathcal{B}x_{2n+1}\}$ also converges to the same point z.

Now suppose that $\mathcal{A}$ is continuous. Then $\{\mathcal{A}\mathcal{A}x_{2n}\}$and $\{\mathcal{A}\mathcal{S}x_{2n}\}$ converges to $\mathcal{A}z$ as $n \to \infty$. Since the mappings $\mathcal{A}$ and $\mathcal{S}$ are compatible in $\mathfrak{V}$, it follows from the Proposition 2.2 that $\{\mathcal{S}\mathcal{A}x_{2n}\}$ converges to $\mathcal{A}z$ as $n \to \infty$.

Now we claim that $z = \mathcal{A}z$. For this put $u = \mathcal{A}x_{2n}$ and $v = x_{2n+1}$ in (C2), we get

$$d^3(\mathcal{S}\mathcal{A}x_{2n}, \mathcal{T}x_{2n+1})$$

$$\leq \psi \left\{ \begin{array}{c} d^2(\mathcal{A}\mathcal{A}x_{2n}, \mathcal{S}\mathcal{A}x_{2n}) d(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}), \\ d(\mathcal{A}\mathcal{A}x_{2n}, \mathcal{S}\mathcal{A}x_{2n}) d^2(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}), \\ d(\mathcal{A}\mathcal{A}x_{2n}, \mathcal{S}\mathcal{A}x_{2n}) d(\mathcal{A}\mathcal{A}x_{2n}, \mathcal{T}x_{2n+1}) d(\mathcal{B}x_{2n+1}, \mathcal{S}\mathcal{A}x_{2n}), \\ d(\mathcal{A}\mathcal{A}x_{2n}, \mathcal{T}x_{2n+1}) d(\mathcal{B}x_{2n+1}, \mathcal{S}\mathcal{A}x_{2n}) d(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}) \end{array} \right\}$$

$$- \emptyset \{m(\mathcal{A}\mathcal{A}x_{2n}, \mathcal{B}x_{2n+1})\},$$

where $$m(\mathcal{A}\mathcal{A}x_{2n}, \mathcal{B}x_{2n+1}) = max \left\{ \begin{array}{c} d^2(\mathcal{A}\mathcal{A}x_{2n}, \mathcal{B}x_{2n+1}), \\ d(\mathcal{A}\mathcal{A}x_{2n}, \mathcal{S}\mathcal{A}x_{2n}) d(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}), \\ d(\mathcal{A}\mathcal{A}x_{2n}, \mathcal{T}x_{2n+1}) d(\mathcal{B}x_{2n+1}, \mathcal{S}\mathcal{A}x_{2n}), \\ \frac{1}{2}\left[\begin{array}{l} d(\mathcal{A}\mathcal{A}x_{2n}, \mathcal{S}\mathcal{A}x_{2n}) d(\mathcal{A}\mathcal{A}x_{2n}, \mathcal{T}x_{2n+1}) \\ +d(\mathcal{B}x_{2n+1}, \mathcal{S}\mathcal{A}x_{2n}) d(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}) \end{array} \right] \end{array} \right\}$$

or $$d^3(\mathcal{A}z, z) \leq \psi \left\{ \begin{array}{c} d^2(\mathcal{A}z, \mathcal{A}z) d(z, z), \\ d(\mathcal{A}z, \mathcal{A}z) d^2(z, z), \\ d(\mathcal{A}z, \mathcal{A}z) d(\mathcal{A}z, z) d(z, \mathcal{A}z), \\ d(\mathcal{A}z, z) d(z, \mathcal{A}z) d(z, z) \end{array} \right\} - \emptyset \{m(\mathcal{A}z, z)\},$$

$$\text{where} \quad m(\mathcal{A}z, z) = max\left\{\begin{matrix} d^2(\mathcal{A}z, z), \\ d(\mathcal{A}z, z)d(z, z), \\ d(\mathcal{A}z, z)d(z, \mathcal{A}z), \\ \frac{1}{2}\begin{bmatrix} d(\mathcal{A}z, \mathcal{A}z)d(\mathcal{A}z, z) \\ +d(z, \mathcal{A}z)d(z, z) \end{bmatrix} \end{matrix}\right\} = d^2(\mathcal{A}z, z)$$

Therefore, we have

$d^3(\mathcal{A}z, z) \le \psi\{0,0,0,0\} - \emptyset\big(d^2(\mathcal{A}z, z)\big)$, using property of ψ and $\emptyset$, we have $\mathcal{A}z = z$.

Now we claim that $z = \mathcal{S}z$. For this put $u = z$ and $v = x_{2n+1}$ in (C2), we get

$$d^3(\mathcal{S}z, \mathcal{T}x_{2n+1})$$

$$\le \psi\left\{\begin{matrix} d^2(\mathcal{A}z, \mathcal{S}z)d(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}), \\ d(\mathcal{A}z, \mathcal{S}z)d^2(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}), \\ d(\mathcal{A}z, \mathcal{S}z)d(\mathcal{A}z, \mathcal{T}x_{2n+1})d(\mathcal{B}x_{2n+1}, \mathcal{S}z), \\ d(\mathcal{A}z, \mathcal{T}x_{2n+1})d(\mathcal{B}x_{2n+1}, \mathcal{S}z)d(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}) \end{matrix}\right\}$$

$$- \emptyset\{m(\mathcal{A}z, \mathcal{B}x_{2n+1})\},$$

$$\text{where} \quad m(\mathcal{A}z, \mathcal{B}x_{2n+1}) = max\left\{\begin{matrix} d^2(\mathcal{A}z, \mathcal{B}x_{2n+1}), \\ d(\mathcal{A}z, \mathcal{S}\mathcal{A}x_{2n})d(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}), \\ d(\mathcal{A}z, \mathcal{T}x_{2n+1})d(\mathcal{B}x_{2n+1}, \mathcal{S}z), \\ \frac{1}{2}\begin{bmatrix} d(\mathcal{A}z, \mathcal{S}z)d(\mathcal{A}z, \mathcal{T}x_{2n+1}) \\ +d(\mathcal{B}x_{2n+1}, \mathcal{S}z)d(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}) \end{bmatrix} \end{matrix}\right\}$$

$$\text{or} \quad d^3(\mathcal{S}z, z) \le \psi\left\{\begin{matrix} d^2(\mathcal{A}z, \mathcal{S}z)d(z, z), \\ d(\mathcal{A}z, \mathcal{S}z)d^2(z, z), \\ d(\mathcal{A}z, \mathcal{S}z)d(\mathcal{A}z, z)d(z, \mathcal{S}z), \\ d(\mathcal{A}z, z)d(z, \mathcal{S}z)d(z, z) \end{matrix}\right\} - \emptyset\{m(\mathcal{A}z, z)\},$$

$$\text{where} \quad m(\mathcal{A}z, z) = max\left\{\begin{matrix} d^2(\mathcal{A}z, z), \\ d(\mathcal{A}z, \mathcal{S}z)d(z, z), \\ d(\mathcal{A}z, z)d(z, \mathcal{S}z), \\ \frac{1}{2}\begin{bmatrix} d(\mathcal{A}z, \mathcal{S}z)d(\mathcal{A}z, z) \\ +d(z, \mathcal{S}z)d(z, z) \end{bmatrix} \end{matrix}\right\} = 0.$$

Therefore, we have

$d^3(\mathcal{S}z, z) \le \psi\{0,0,0,0\} - \emptyset(0)$,using property of ψ and $\emptyset$, we have $d^3(\mathcal{S}z, z) = 0$.

This implies that $\mathcal{S}z = z$. Since $\mathcal{S}(\mathfrak{V}) \subset \mathcal{B}(\mathfrak{V})$ and hence there exists a point $p \in \mathfrak{V}$ such that $z = \mathcal{S}z = \mathcal{B}p$.

We claim that $z = \mathcal{T}u$. To prove this we put $u = z$ and $v = p$ in (C2), we get

$$d^3(\mathcal{S}z,\mathcal{T}\wp) \leq \psi \left\{ \begin{matrix} d^2(\mathcal{A}z,\mathcal{S}z)d(\mathcal{B}\wp,\mathcal{T}\wp), \\ d(\mathcal{A}z,\mathcal{S}z)d^2(\mathcal{B}\wp,\mathcal{T}\wp), \\ d(\mathcal{A}z,\mathcal{S}z)d(\mathcal{A}z,\mathcal{T}\wp)d(\mathcal{B}\wp,\mathcal{S}z), \\ d(\mathcal{A}z,\mathcal{T}\wp)d(\mathcal{B}\wp,\mathcal{S}z)d(\mathcal{B}\wp,\mathcal{T}\wp) \end{matrix} \right\} - \emptyset\,\{m(\mathcal{A}z,\mathcal{B}\wp)\},$$

where $$m(\mathcal{A}z,\mathcal{B}\wp) = max \left\{ \begin{matrix} d^2(\mathcal{A}z,\mathcal{B}\wp), \\ d(\mathcal{A}z,\mathcal{S}z)d(\mathcal{B}\wp,\mathcal{T}\wp), \\ d(\mathcal{A}z,\mathcal{T}\wp)d(\mathcal{B}\wp,\mathcal{S}z), \\ \frac{1}{2}\left[\begin{matrix} d(\mathcal{A}z,\mathcal{S}z)d(\mathcal{A}z,\mathcal{T}\wp) \\ +d(\mathcal{B}\wp,\mathcal{S}z)d(\mathcal{B}\wp,\mathcal{T}\wp) \end{matrix} \right] \end{matrix} \right\} = 0,$$

i.e., $$m(\mathcal{A}z,\mathcal{B}\wp) = max \left\{ \begin{matrix} d^2(z,z), \\ d(z,z)d(z,\mathcal{T}\wp), \\ d(z,\mathcal{T}\wp)d(z,z), \\ \frac{1}{2}\left[\begin{matrix} d(z,z)d(\mathcal{A}z,\mathcal{T}\wp) \\ +d(z,z)d(z,\mathcal{T}\wp) \end{matrix} \right] \end{matrix} \right\} = 0$$

On simplification, and using property of ψ and $\emptyset$, we have

$$d^3(z,\mathcal{T}\wp) \leq \psi \left\{ \begin{matrix} d^2(z,z)d(z,\mathcal{T}\wp), \\ d(z,z)d^2(z,\mathcal{T}\wp), \\ d(z,z)d(z,\mathcal{T}\wp)d(z,z), \\ d(z,\mathcal{T}\wp)d(z,z)d(z,\mathcal{T}\wp) \end{matrix} \right\} - \emptyset(0).$$

This implies that $z = \mathcal{T}\wp$. Since $(\mathcal{B},\mathcal{T})$ is compatible in $\mathfrak{V}$ and $\mathcal{B}\wp = \mathcal{T}\wp = z$, by Proposition 2.1, we have $\mathcal{B}\mathcal{T}\wp = \mathcal{T}\mathcal{B}\wp$ and hence $\mathcal{B}z = \mathcal{B}\mathcal{T}\wp = \mathcal{T}\mathcal{B}\wp = \mathcal{T}z$. Also, we have

$$d^3(\mathcal{S}z,\mathcal{T}z) \leq \psi \left\{ \begin{matrix} d^2(\mathcal{A}z,\mathcal{S}z)d(\mathcal{B}z,\mathcal{T}z), \\ d(\mathcal{A}z,\mathcal{S}z)d^2(\mathcal{B}z,\mathcal{T}z), \\ d(\mathcal{A}z,\mathcal{S}z)d(\mathcal{A}z,\mathcal{T}z)d(\mathcal{B}z,\mathcal{S}z), \\ d(\mathcal{A}z,\mathcal{T}z)d(\mathcal{B}z,\mathcal{S}z)d(\mathcal{B}z,\mathcal{T}z) \end{matrix} \right\} - \emptyset(m(\mathcal{A}z,\mathcal{B}z)),$$

where $$m(\mathcal{A}z,\mathcal{B}z) = max \left\{ \begin{matrix} d^2(\mathcal{A}z,\mathcal{B}z), \\ d(\mathcal{A}z,\mathcal{S}z)d(\mathcal{B}z,\mathcal{T}z), \\ d(\mathcal{A}z,\mathcal{T}z)d(\mathcal{B}z,\mathcal{S}z), \\ \frac{1}{2}\left[\begin{matrix} d(\mathcal{A}z,\mathcal{S}z)d(\mathcal{A}z,\mathcal{T}z) \\ +d(\mathcal{B}z,\mathcal{S}z)d(\mathcal{B}z,\mathcal{T}z) \end{matrix} \right] \end{matrix} \right\} = d^2(z,\mathcal{B}z).$$

Therefore, we obtain

$d^3(z,\mathcal{B}z) \leq \psi\{0,0,0,0\} - \emptyset(d^2(z,\mathcal{B}z))$., using property of ψ and $\emptyset$, we have

i.e., $d^2(z,\mathcal{B}z) \leq 0$.

This implies that $z = \mathcal{B}z$. Hence $z = \mathcal{B}z = \mathcal{T}z = \mathcal{A}z = \mathcal{S}z$. Therefore, z is a common fixed point of $\mathcal{S}, \mathcal{T}, \mathcal{A}$ and $\mathcal{B}$.

Similarly, one can also complete the proof when $\mathcal{B}$ is continuous.

Next, suppose that $\mathcal{S}$ is continuous.

Then $\{\mathcal{S}\mathcal{S}x_{2n}\}$ and $\{\mathcal{S}\mathcal{A}x_{2n}\}$ converges to $\mathcal{S}z$ as $n \to \infty$. Since the mappings $\mathcal{A}$ and $\mathcal{S}$ are compatible on $\mathfrak{V}$, it follows from the proposition 2.2 that $\{\mathcal{A}\mathcal{S}x_{2n}\}$ converges to $\mathcal{S}z$ as $n \to \infty$.

Now we claim that $z = \mathcal{S}z$. For this put $u = \mathcal{S}x_{2n}$ and $v = x_{2n+1}$ in (C2), we get

$$d^3(\mathcal{S}\mathcal{S}x_{2n}, \mathcal{T}x_{2n+1})$$

$$\leq \psi \left\{ \begin{array}{c} d^2(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{S}\mathcal{S}x_{2n})d(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}), \\ d(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{S}\mathcal{S}x_{2n})d^2(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}), \\ d(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{S}\mathcal{S}x_{2n})d(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{T}x_{2n+1})d(\mathcal{B}x_{2n+1}, \mathcal{S}\mathcal{S}x_{2n}), \\ d(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{T}x_{2n+1})d(\mathcal{B}x_{2n+1}, \mathcal{S}\mathcal{S}x_{2n})d(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}) \end{array} \right\}$$

$$- \emptyset \{m(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{B}x_{2n+1})\},$$

where $$m(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{B}x_{2n+1}) = max \left\{ \begin{array}{c} d^2(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{B}x_{2n+1}), \\ d(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{S}\mathcal{S}x_{2n})d(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}), \\ d(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{T}x_{2n+1})d(\mathcal{B}x_{2n+1}, \mathcal{S}\mathcal{S}x_{2n}), \\ \frac{1}{2}\left[\begin{array}{c} d(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{S}\mathcal{S}x_{2n})d(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{T}x_{2n+1}) \\ +d(\mathcal{B}x_{2n+1}, \mathcal{S}\mathcal{S}x_{2n})d(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}) \end{array} \right] \end{array} \right\}$$

Now proceeding limit as $n \to \infty$ and using the property of ψ and $\emptyset$, we have

$$d^3(\mathcal{S}z, z) \leq \psi \left\{ \begin{array}{c} d^2(\mathcal{S}z, \mathcal{S}z)d(z, z), \\ d(\mathcal{S}z, \mathcal{S}z)d^2(z, z), \\ d(\mathcal{S}z, \mathcal{S}z)d(\mathcal{S}z, z)d(z, \mathcal{S}z), \\ d(\mathcal{S}z, z)d(z, \mathcal{S}z)d(z, z) \end{array} \right\} - \emptyset \{m(\mathcal{S}z, z)\},$$

where $$m(\mathcal{S}z, z) = max \left\{ \begin{array}{c} d^2(\mathcal{S}z, z), \\ d(\mathcal{S}z, \mathcal{S}z)d(z, z), \\ d(\mathcal{S}z, z)d(z, \mathcal{S}z), \\ \frac{1}{2}\left[\begin{array}{c} d(\mathcal{S}z, \mathcal{S}z)d(\mathcal{S}z, z) \\ +d(z, \mathcal{S}z)d(z, z) \end{array} \right] \end{array} \right\} = d^2(\mathcal{S}z, z).$$

Therefore, we have $d^3(\mathcal{S}z, z) \leq \psi\{0,0,0,0\} - \emptyset\left(d^2(\mathcal{S}z, z)\right)$,using property of ψ and $\emptyset$, we have $d^3(\mathcal{S}z, z) = 0$. This implies that $\mathcal{S}z = z$. Since $\mathcal{S}(\mathfrak{V}) \subset \mathcal{B}(\mathfrak{V})$ and hence there exists a point $q \in \mathfrak{V}$ such that $z = \mathcal{S}z = \mathcal{B}q$.

We claim that $z = \mathcal{T}q$. To prove this, we put $u = \mathcal{S}x_{2n}$ and $v = q$ in (C2) we get

$$d^3(\mathcal{S}\mathcal{S}x_{2n}, \mathcal{T}q)$$

$$\leq \psi \left\{ \begin{matrix} d^2(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{S}\mathcal{S}x_{2n})d(\mathcal{B}q, \mathcal{T}q), \\ d(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{S}\mathcal{S}x_{2n})d^2(\mathcal{B}q, \mathcal{T}q), \\ d(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{S}\mathcal{S}x_{2n})d(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{T}q)d(\mathcal{B}x_{2n+1}, \mathcal{S}\mathcal{S}x_{2n}), \\ d(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{T}q)d(\mathcal{B}x_{2n+1}, \mathcal{S}\mathcal{S}x_{2n})d(\mathcal{B}q, \mathcal{T}q) \end{matrix} \right\}$$

$$- \emptyset \{m(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{B}q)\},$$

where $$m(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{B}q) = max \left\{ \begin{matrix} d^2(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{B}q), \\ d(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{S}\mathcal{S}x_{2n})d(\mathcal{B}q, \mathcal{T}q), \\ d(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{T}q)d(\mathcal{B}q, \mathcal{S}\mathcal{S}x_{2n}), \\ \frac{1}{2}\begin{bmatrix} d(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{S}\mathcal{S}x_{2n})d(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{T}q) \\ +d(\mathcal{B}q, \mathcal{S}\mathcal{S}x_{2n})d(\mathcal{B}q, \mathcal{T}q) \end{bmatrix} \end{matrix} \right\}$$

i.e., $$m(z, \mathcal{B}q) = max \left\{ \begin{matrix} d^2(z, z), \\ d(z, z)d(z, \mathcal{T}q), \\ d(z, \mathcal{T}q)d(z, z), \\ \frac{1}{2}\begin{bmatrix} d(z, z)d(z, \mathcal{T}q) \\ +d(z, z)d(z, \mathcal{T}q) \end{bmatrix} \end{matrix} \right\} = 0$$

Therefore, we get

$$d^3(z, \mathcal{T}q) \leq \psi \left\{ \begin{matrix} d^2(z, z)d(z, \mathcal{T}q), \\ d(z, z)d^2(z, \mathcal{T}q), \\ d(z, z)d(z, \mathcal{T}q)d(z, z), \\ d(z, \mathcal{T}q)d(z, z)d(z, \mathcal{T}q) \end{matrix} \right\} - \emptyset(0).$$

Using the property of ψ and $\emptyset$, we have $z = \mathcal{T}q$. Since $(\mathcal{B}, \mathcal{T})$ is a compatible pair of mappings, so $\mathcal{B}q = \mathcal{T}q = z$ and by using Proposition 2.1 we have $\mathcal{B}\mathcal{T}q = \mathcal{T}\mathcal{B}q$ and hence $\mathcal{B}z = \mathcal{B}\mathcal{T}q = \mathcal{T}\mathcal{B}q = \mathcal{T}z$. On putting $u = x_{2n}$ and $v = z$ in (C2), we have

$$d^3(\mathcal{S}x_{2n}, \mathcal{T}z) \leq \psi \left\{ \begin{matrix} d^2(\mathcal{A}x_{2n}, \mathcal{S}x_{2n})d(\mathcal{B}z, \mathcal{T}z), \\ d(\mathcal{A}x_{2n}, \mathcal{S}x_{2n})d^2(\mathcal{B}z, \mathcal{T}z), \\ d(\mathcal{A}x_{2n}, \mathcal{S}x_{2n})d(\mathcal{A}x_{2n}, \mathcal{T}z)d(\mathcal{B}z, \mathcal{S}x_{2n}), \\ d(\mathcal{A}x_{2n}, \mathcal{T}z)d(\mathcal{B}z, \mathcal{S}x_{2n})d(\mathcal{B}z, \mathcal{T}z) \end{matrix} \right\} - \emptyset \{m(\mathcal{A}x_{2n}, \mathcal{B}z)\},$$

where $$m(\mathcal{A}x_{2n}, \mathcal{B}z) = max \left\{ \begin{matrix} d^2(\mathcal{A}x_{2n}, \mathcal{B}z), \\ d(\mathcal{A}x_{2n}, \mathcal{S}x_{2n})d(\mathcal{B}z, \mathcal{T}z), \\ d(\mathcal{A}x_{2n}, \mathcal{T}z)d(\mathcal{B}z, \mathcal{S}x_{2n}), \\ \frac{1}{2}\begin{bmatrix} d(\mathcal{A}x_{2n}, \mathcal{S}x_{2n})d(\mathcal{A}x_{2n}, \mathcal{T}z) \\ +d(\mathcal{B}z, \mathcal{S}x_{2n})d(\mathcal{B}z, \mathcal{T}z) \end{bmatrix} \end{matrix} \right\} = d^2(z, \mathcal{T}z)$$

Proceeding limit as $n \to \infty$,we get

$d^3(z,\mathcal{T}z) \leq \psi\{0,0,0,0\} - \emptyset\{d^2(z,\mathcal{T}z)\}$.

Using the property of ψ and $\emptyset$, we have $z = \mathcal{T}z$. Since $\mathcal{T}(\mathfrak{V}) \subset \mathcal{A}(\mathfrak{V})$, therefore there exists a point $w \in \mathfrak{V}$ such that $z = \mathcal{T}z = \mathcal{A}w$.

We claim that $z = \mathcal{S}w$. On putting $u = w$ and $v = z$ in (C2) we get

$$d^3(\mathcal{S}w,\mathcal{T}z) \leq \psi\left\{\begin{matrix} d^2(\mathcal{A}w,\mathcal{S}w)d(\mathcal{B}z,\mathcal{T}z), \\ d(\mathcal{A}w,\mathcal{S}w)d^2(\mathcal{B}z,\mathcal{T}z), \\ d(\mathcal{A}w,\mathcal{S}w)d(\mathcal{A}w,\mathcal{T}z)d(\mathcal{B}z,\mathcal{S}w), \\ d(\mathcal{A}w,\mathcal{T}z)d(\mathcal{B}z,\mathcal{S}w)d(\mathcal{B}z,\mathcal{T}z) \end{matrix}\right\} - \emptyset\,\{m(\mathcal{A}w,\mathcal{B}z)\},$$

where $$m(\mathcal{A}w,\mathcal{B}z) = max\left\{\begin{matrix} d^2(\mathcal{A}w,\mathcal{B}z), \\ d(\mathcal{A}w,\mathcal{S}w)d(\mathcal{B}z,\mathcal{T}z), \\ d(\mathcal{A}w,Tz)d(\mathcal{B}z,\mathcal{S}w), \\ \frac{1}{2}\begin{bmatrix} d(\mathcal{A}w,\mathcal{S}w)d(\mathcal{A}w,\mathcal{T}z) \\ +d(\mathcal{B}z,\mathcal{S}w)d(\mathcal{B}z,\mathcal{T}z) \end{bmatrix} \end{matrix}\right\} = 0.$$

i.e., $$m(\mathcal{A}w,\mathcal{B}z) = max\left\{\begin{matrix} d^2(z,z), \\ d(z,\mathcal{S}w)d(Tz,Tz), \\ d(z,z)d(z,\mathcal{S}w), \\ \frac{1}{2}\begin{bmatrix} d(z,\mathcal{S}w)d(z,z) \\ +d(z,\mathcal{S}w)d(Tz,Tz) \end{bmatrix} \end{matrix}\right\} = 0$$

Therefore, $$d^3(\mathcal{S}w,z) \leq \psi\left\{\begin{matrix} d^2(z,\mathcal{S}w)d(z,z), \\ d(z,\mathcal{S}w)d^2(z,z), \\ d(z,\mathcal{S}w)d(z,z)d(z,\mathcal{S}w), \\ d(z,z)d(z,\mathcal{S}w)d(z,z) \end{matrix}\right\} - \emptyset\,\{0\},$$

This implies that $\mathcal{S}w = z$. Since pair $(\mathcal{S},\mathcal{A})$ is compatible on $\mathfrak{V}$, so, $\mathcal{S}w = \mathcal{A}w = z$ and by Proposition 2.1, we have $\mathcal{A}\mathcal{S}w = \mathcal{S}\mathcal{A}w$. Thus $\mathcal{A}z = \mathcal{A}\mathcal{S}w = \mathcal{S}\mathcal{A}w = \mathcal{S}z$.

i.e., $z = \mathcal{A}z = \mathcal{S}z = \mathcal{B}z = \mathcal{T}z$. Therefore, z is a common fixed point of $\mathcal{S},\mathcal{T},\mathcal{A}$ and $\mathcal{B}$.

Similarly, we can complete the proof when $\mathcal{T}$ is continuous.

Uniqueness: Suppose $z \neq w$ be two common fixed points of $\mathcal{S},\mathcal{T},\mathcal{A}$ and $\mathcal{B}$.

Put $u = z$ and $v = w$ in (C2), we get

$d^3(\mathcal{S}z,\mathcal{T}w) \leq \psi\{0,0,0,0\} - \emptyset(m(\mathcal{A}z,\mathcal{B}w))$

$d^3(\mathcal{S}z,\mathcal{T}w) \leq \psi\{0,0,00\} - \emptyset(d^2(\mathcal{S}z,\mathcal{T}w))$

On simplification, using the property of ψ and $\emptyset$, we have we have $d^2(z, w) = 0$

$i.e., \ z = w.$

This completes the proof.

If we put $\psi(t) = 0$ for all t in Theorem 2.1, we have the following result:

Corollary 2.1 Let $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ be four mappings of a complete metric space $(\mathfrak{B}, d)$ into itself satisfying (C1), (C2) and the following condition:

$$d^2(\mathcal{S}x, \mathcal{T}y) \leq 0 - \emptyset(m(\mathcal{A}x, \mathcal{B}y)), \text{ for all } x, y \in \mathfrak{B}$$

where $m(\mathcal{A}x, \mathcal{B}y) = max \left\{ \begin{array}{c} d^2(\mathcal{A}x, \mathcal{B}y), \\ d(\mathcal{A}x, \mathcal{S}x)d(\mathcal{B}y, \mathcal{T}y), \\ d(\mathcal{A}x, \mathcal{T}y)d(\mathcal{B}y, \mathcal{S}x), \\ \frac{1}{2}\left[\begin{array}{c} d(\mathcal{A}x, \mathcal{S}x)d(\mathcal{A}x, \mathcal{T}y) \\ +d(\mathcal{B}y, \mathcal{S}x)d(\mathcal{B}y, \mathcal{T}y) \end{array} \right] \end{array} \right\}$

$\emptyset: [0, \infty) \rightarrow [0, \infty)$ is a continuous function with $\emptyset(t) = 0 \Leftrightarrow t = 0$ and $\emptyset(t) > 0$ for each $t > 0$. Assume that the pairs $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ are compatible. Then $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ has a unique fixed point in $\mathfrak{B}$.

3. VARIANTS OF COMPATIBLE MAPPINGS AND FIXED POINTS

Fixed point theorems are statements containing sufficient conditions that ensure the existence of a fixed point. Therefore, one of the central concerns in fixed point theory is to find a minimal set of sufficient conditions which guarantee a fixed point or a common fixed point as the case may be. It was a landmark in the fixed point theory literature when the notion of commutativity mappings was used by Jungck [6] to obtain a common fixed point theorem for a pair of mappings by using a constructive procedure of sequence of iterates. The essence of Jungck's theorem has been used by several workers to obtain interesting common fixed point theorems for both commuting and non commuting pairs of mappings satisfying contractive type conditions. The constructive technique of Jungck's theorem has been further improved and extended by various researchers to establish common fixed point theorems for three mappings, four mappings and sequence of mappings. Common fixed point theorems for contractive type mappings necessarily require a commutativity condition, a condition on the ranges of the mappings, continuity of one or more mappings besides a contractive condition.

In 1993, Jungck et al. [10] introduced the notion of compatible mappings of type(A) as follows:

Definition 3.1 [10] Two self mappings f and g of a metric space $(\mathfrak{V}, d)$ are called compatible of type(A) if $lim_n d(ffx_n, gfx_n) = 0$ and $lim_n d(ggx_n, fgx_n) = 0$,
whenever$\{x_n\}$ is a sequence in $\mathfrak{V}$ such that $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$.

In 1995, Pathak et al. [13] introduced the notion of compatible mappings of type(P) as follows:

Definition 3.2[13]Two self mappings f and g of a metric space $(\mathfrak{V}, d)$ are called compatible of type(P) if $lim_n d(ffx_n, ggx_n) = 0$,whenever$\{x_n\}$ is a sequence in $\mathfrak{V}$ such that $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$.

In 1998, Pant [17] defined the notion of reciprocally continuous mappings. In fact, it is the generalization of continuous mappings.

Dentition 3.3[17] Two self mappings f and g of a metric space $(\mathfrak{V}, d)$are called reciprocally continuous if $lim_n fgx_n = ft$ and $lim_n gfx_n = gt$, whenever $\{x_n\}$ is a sequence in $\mathfrak{V}$ such that $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$.

If f and g are both continuous, then maps are reciprocally continuous, but the converse need not be true.

In 2001, Sahu et al. [22] introduced the notion of intimate mappings in metric spaces. In fact, it is the generalization of compatible mappings of type (A).

Definition 3.4[22] Let f and g are two mappings of a metric space $(\mathfrak{V}, d)$ into itself. Then f and g are said to be:

(1) g-intimate mappings if $\alpha d(gfx_n, gx_n) \leq \alpha d(ffx_n, fx_n)$, where $\{x_n\}$ is a sequence in $\mathfrak{V}$ such that $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$ and α = $lim\ sup\ or\ lim\ inf$.

(2) f-intimate mappings if $\alpha d(fgx_n, fx_n) \leq \alpha d(ggx_n, gx_n)$, where $\{x_n\}$ is a sequence in $\mathfrak{V}$ such that $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$ and $\alpha = lim\ sup\ or\ lim\ inf$.

In 2004, Rohan et al. [20] introduced the concept of compatible mappings of type (R) as follows:

Definition 3.5[20] Two self-mappings f and g of a metric space $(\mathfrak{V}, d)$ are called compatible of type (R) if $lim_n d(fgx_n, gfx_n) = 0$ and $lim_n d(ffx_n, ggx_n) = 0$, whenever $\{x_n\}$ is a sequence in $\mathfrak{V}$ such that $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$.

In 2007, Singh and Singh [23] introduced the concept of compatible mappings of type (E) by rearranging terms of compatible mappings of type (P) and compatible mappings

Definition 3.6 [23] Two self-mappings f and g of a metric space $(\mathfrak{V}, d)$ are called compatible of type (E) if $lim_n ffx_n = lim_n fgx_n = gt$ and $lim_n ggx_n = lim_n gfx_n = ft$, whenever $\{x_n\}$ is a sequence in $\mathfrak{V}$ such that $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$.

In 2014, Jha et al. [11] introduced the concept of compatible mappings of type (K) by modification in compatible mappings of type (P) in a metric space as follows:

Definition 3.7[11] Two self-mappings f and g of a metric space $(\mathfrak{V}, d)$ are called compatible of type (K) if $lim_n d(ffx_n, gt) = 0$ and $lim_n d(ggx_n, ft) = 0$,
whenever $\{x_n\}$ is a sequence in $\mathfrak{V}$ such that $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$.

We describe the relationship among compatible maps and its variants in metric spaces which are useful for proving our main results.

Remark 3.1 One can note that compatible mapping of type (R) is compatible mapping as well as compatible mappings of type (P).

Proposition 3.1 [23] Suppose f and g be compatible mappings of type (E) of a metric space $(\mathfrak{V}, d)$ into itself and one of f and g be continuous. Suppose $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$. Then we have the following:

(a) $ft = gt$ and $lim_n ffx_n = lim_n ggx_n = lim_n fgx_n = lim_n gfx_n$.

(b) If there exists $u \in \mathfrak{V}$ such that $fu = gu = t$, then $fgu = gfu$.

Proposition 3.2 Let f and g be two mappings of a metric space $(\mathfrak{V}, d)$ into itself. If f and g are compatible mappings of type (A), then f and g are f-intimate and g-intimate.

Remark 3.2 If a pair (f, g) is f-intimate or g-intimate then it need not be necessarily compatible of type (A).

Proposition 3.3 [22] Let f and g be two mappings of a metric space $(\mathfrak{V}, d)$ into itself. Assume that f and g are g-intimate and $ft = gt = q \in \mathfrak{V}$. Then $d(gq, q) \leq d(fq, q)$.

We now prove some results in metric spaces related to compatible mappings of type (K), type (R), type (E) and intimate mappings that satisfy generalized $\psi - \emptyset -$weak contraction condition that involves cubic and quadratic terms of distance function.

Theorem 3.1 Let $\mathcal{S}, \mathcal{T}, \mathcal{A}$ and $\mathcal{B}$ are four self mappings of a complete metric space $(\mathfrak{V}, d)$ satisfying (C1) and (C2) and the following conditions:

(3.1) the pairs $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ are reciprocally continuous,

(3.2) the pairs $(\mathcal{A}, \mathcal{S})$and $(\mathcal{B}, \mathcal{T})$ are compatible of type (K).

Then $z = \mathcal{A}z = \mathcal{S}z = \mathcal{B}z = \mathcal{T}z$, and z is unique in $\mathfrak{V}$.

Proof. From the Theorem 2.1, we conclude the sequence $\{y_n\}$ is a Cauchy sequence in $\mathfrak{V}$, but $(\mathfrak{V}, d)$ is a complete metric space, therefore, $\{y_n\}$ converges to a point z in $\mathfrak{V}$ as $n \to \infty$. Consequently, the subsequences $\{\mathcal{S}x_{2n}\}, \{\mathcal{A}x_{2n}\}, \{\mathcal{T}x_{2n+1}\}$ and $\{\mathcal{B}x_{2n+1}\}$ also converges to the same point z. Now Since the pairs $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ are compatible of type(K), we have $\mathcal{A}\mathcal{A}x_{2n} \to \mathcal{S}z,\ \mathcal{S}\mathcal{S}x_{2n} \to \mathcal{A}z$ and $\mathcal{B}\mathcal{B}x_{2n} \to \mathcal{T}z,\ \mathcal{T}\mathcal{T}x_{2n} \to \mathcal{B}z$ as $n \to \infty$.

Now we claim that $\mathcal{B}z = \mathcal{A}z$. For this put $u = \mathcal{S}x_{2n}$ and $v = \mathcal{T}x_{2n+1}$ in (C2) we get

$$
\begin{aligned}
&d^3(\mathcal{S}\mathcal{S}x_{2n}, \mathcal{T}\mathcal{T}x_{2n+1}) \\
&\qquad \leq \psi \left\{ \begin{array}{c} d^2(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{S}\mathcal{S}x_{2n}) d(\mathcal{B}\mathcal{T}x_{2n+1}, \mathcal{T}\mathcal{T}x_{2n+1}), \\ d(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{S}\mathcal{S}x_{2n}) d^2(\mathcal{B}\mathcal{T}x_{2n+1}, \mathcal{T}\mathcal{T}x_{2n+1}), \\ d(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{S}\mathcal{S}x_{2n}) d(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{T}\mathcal{T}x_{2n+1}) d(\mathcal{B}\mathcal{T}x_{2n+1}, \mathcal{S}\mathcal{S}x_{2n}), \\ d(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{T}\mathcal{T}x_{2n+1}) d(\mathcal{B}\mathcal{T}x_{2n+1}, \mathcal{S}\mathcal{S}x_{2n}) d(\mathcal{B}\mathcal{T}x_{2n+1}, \mathcal{T}\mathcal{T}x_{2n+1}) \end{array} \right\} \\
&\qquad - \emptyset \{m(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{B}\mathcal{T}x_{2n+1})\},
\end{aligned}
$$

Where $m(\mathcal{AS}x_{2n}, \mathcal{BT}x_{2n+1}) = max \left\{ \begin{matrix} d^2(\mathcal{AS}x_{2n}, \mathcal{BT}x_{2n+1}), \\ d(\mathcal{AS}x_{2n}, \mathcal{SS}x_{2n})d(\mathcal{BT}x_{2n+1}, \mathcal{TT}x_{2n+1}), \\ d(\mathcal{AS}x_{2n}, \mathcal{TT}x_{2n+1})d(\mathcal{BT}x_{2n+1}, \mathcal{SS}x_{2n}), \\ \frac{1}{2}\begin{bmatrix} d(\mathcal{AS}x_{2n}, \mathcal{SS}x_{2n})d(\mathcal{AS}x_{2n}, \mathcal{TT}x_{2n+1}) \\ +d(\mathcal{BT}x_{2n+1}, \mathcal{SS}x_{2n})d(\mathcal{BT}x_{2n+1}, \mathcal{TT}x_{2n+1}) \end{bmatrix} \end{matrix} \right\}$

Letting $n \to \infty$ and using reciprocal continuity of the pairs $(\mathcal{A}, \mathcal{S})$and $(\mathcal{B}, \mathcal{T})$, we have $d^3(\mathcal{B}z, \mathcal{A}z.) \leq \psi\{0,0,0,0\} - \emptyset(d^2(\mathcal{B}z, \mathcal{A}z)$, using property of ψ and $\emptyset$, we have $d^3(\mathcal{B}z, \mathcal{A}z) = 0$.This implies that $\mathcal{B}z = \mathcal{A}z$.

Next, we claim that $\mathcal{S}z = \mathcal{B}z$. On putting $u = z$ and $v = \mathcal{T}x_{2n+1}$ in (C2) we get

$$d^3(\mathcal{S}z, \mathcal{TT}x_{2n+1})$$

$$\leq \psi \left\{ \begin{matrix} d^2(\mathcal{A}z, \mathcal{S}z)d(\mathcal{BT}x_{2n+1}, \mathcal{TT}x_{2n+1}), \\ d(\mathcal{A}z, \mathcal{S}z)d^2(\mathcal{BT}x_{2n+1}, \mathcal{TT}x_{2n+1}), \\ d(\mathcal{A}z, \mathcal{S}z)d(\mathcal{A}z, \mathcal{TT}x_{2n+1})d(\mathcal{BT}x_{2n+1}, \mathcal{S}z), \\ d(\mathcal{A}z, \mathcal{TT}x_{2n+1})d(\mathcal{BT}x_{2n+1}, \mathcal{S}z)d(\mathcal{BT}x_{2n+1}, \mathcal{TT}x_{2n+1}) \end{matrix} \right\}$$

$$- \emptyset\, \{m(\mathcal{A}z, \mathcal{BT}x_{2n+1})\},$$

Where $m(\mathcal{A}z, \mathcal{BT}x_{2n+1}) = max \left\{ \begin{matrix} d^2(\mathcal{A}z, \mathcal{BT}x_{2n+1}), \\ d(\mathcal{A}z, \mathcal{S}z)d(\mathcal{BT}x_{2n+1}, \mathcal{TT}x_{2n+1}), \\ d(\mathcal{A}z, \mathcal{TT}x_{2n+1})d(\mathcal{BT}x_{2n+1}, \mathcal{S}z), \\ \frac{1}{2}\begin{bmatrix} d(\mathcal{A}z, \mathcal{S}z)d(\mathcal{A}z, \mathcal{TT}x_{2n+1}) \\ +d(\mathcal{BT}x_{2n+1}, \mathcal{S}z)d(\mathcal{BT}x_{2n+1}, \mathcal{TT}x_{2n+1}) \end{bmatrix} \end{matrix} \right\}$

Letting $n \to \infty$ and using reciprocal continuity of the pairs $(\mathcal{A}, \mathcal{S})$and $(\mathcal{B}, \mathcal{T})$, we have

$$d^3(\mathcal{S}z, \mathcal{B}z) \leq \psi\{0,0,0,0\} - \emptyset(0),$$

using property of ψ and $\emptyset$, we have $d^3(\mathcal{S}z, \mathcal{B}z) = 0$.This implies that $\mathcal{S}z = \mathcal{B}z$.

Now we claim that $\mathcal{S}z = \mathcal{T}z$. On putting $u = z$ and $v = z$ in (C2) we get

$$d^3(\mathcal{S}z, \mathcal{T}z) \leq \psi \left\{ \begin{matrix} d^2(\mathcal{A}z, \mathcal{S}z)d(\mathcal{B}z, \mathcal{T}z), \\ d(\mathcal{A}z, \mathcal{S}z)d^2(\mathcal{B}z, \mathcal{T}z), \\ d(\mathcal{A}z, \mathcal{S}z)d(\mathcal{A}z, \mathcal{T}z)d(\mathcal{B}z, \mathcal{S}z), \\ d(\mathcal{A}z, \mathcal{T}z)d(\mathcal{B}z, \mathcal{S}z)d(\mathcal{B}z, \mathcal{T}z) \end{matrix} \right\} - \emptyset\, \{m(\mathcal{A}z, \mathcal{B}z)\},$$

where $m(\mathcal{A}z, \mathcal{B}z) = max \left\{ \begin{matrix} d^2(\mathcal{A}z, \mathcal{B}z), d(\mathcal{A}z, \mathcal{S}z)d(\mathcal{B}z, \mathcal{T}z), \\ d(\mathcal{A}z, \mathcal{T}z)d(\mathcal{B}z, \mathcal{S}z), \\ \frac{1}{2}\begin{bmatrix} d(\mathcal{A}z, \mathcal{S}z)d(\mathcal{A}z, \mathcal{T}z) \\ +d(\mathcal{B}z, \mathcal{S}z)d(\mathcal{B}z, \mathcal{T}z) \end{bmatrix} \end{matrix} \right\}$

Proceeding limit as $n \to \infty$,we get

$d^3(\mathcal{S}z, \mathcal{T}z) \quad \leq \; \psi\,\{0,0,0,0\} - \emptyset(0).$

Thus $d^3(\mathcal{S}z, \mathcal{T}z) = 0$, implies that $\mathcal{S}z = \mathcal{T}z$.

Now we claim that $z = \mathcal{T}z$. On putting $u = x_{2n}$ and $v = z$ in (C2), we have

$$d^3(\mathcal{S}x_{2n}, \mathcal{T}z) \leq \psi \left\{ \begin{matrix} d^2(\mathcal{A}x_{2n}, \mathcal{S}x_{2n})d(\mathcal{B}z, \mathcal{T}z), \\ d(\mathcal{A}x_{2n}, \mathcal{S}x_{2n})d^2(\mathcal{B}z, \mathcal{T}z), \\ d(\mathcal{A}x_{2n}, \mathcal{S}x_{2n})d(\mathcal{A}x_{2n}, \mathcal{T}z)d(\mathcal{B}z, \mathcal{S}x_{2n}), \\ d(\mathcal{A}x_{2n}, \mathcal{T}z)d(\mathcal{B}z, \mathcal{S}x_{2n})d(\mathcal{B}z, \mathcal{T}z) \end{matrix} \right\} - \emptyset\,\{m(\mathcal{A}x_{2n}, \mathcal{B}z)\},$$

Where $m(\mathcal{A}x_{2n}, \mathcal{B}z) = max \left\{ \begin{matrix} d^2(\mathcal{A}x_{2n}, \mathcal{B}z), \\ d(\mathcal{A}x_{2n}, \mathcal{S}x_{2n})d(\mathcal{B}z, \mathcal{T}z), \\ d(\mathcal{A}x_{2n}, \mathcal{T}z)d(\mathcal{B}z, \mathcal{S}x_{2n}), \\ \frac{1}{2}\left[\begin{matrix} d(\mathcal{A}x_{2n}, \mathcal{S}x_{2n})d(\mathcal{A}x_{2n}, \mathcal{T}z) \\ +d(\mathcal{B}z, \mathcal{S}x_{2n})d(\mathcal{B}z, \mathcal{T}z) \end{matrix} \right] \end{matrix} \right\} = d^2(z, \mathcal{T}z)$

Proceeding limit as $n \to \infty$,weget

$d^3(z, \mathcal{T}z) \leq \; \psi\{0,0,0,0\} - \emptyset\{d^2(z, \mathcal{T}z)\}$.Uniqueness follows easily

Then $z = \mathcal{A}z = \mathcal{S}z = \mathcal{B}z = \mathcal{T}z$, and z is unique in $\mathfrak{V}$.

First, we prove the following theorem for compatible mappings of type (R).

Theorem 3.2 Let $\mathcal{S}, \mathcal{T}, \mathcal{A}$ and $\mathcal{B}$ are four self mappings of a complete metric space $(\mathfrak{V}, d)$ satisfying (C1) and (C2) and the following conditions:

(3.3) One of $\mathcal{S}, \mathcal{T}, \mathcal{A}$ and $\mathcal{B}$ is continuous.

Assume that the pairs $(\mathcal{A}, \mathcal{S})$and $(\mathcal{B}, \mathcal{T})$ are compatible of type (R). Then $z = \mathcal{A}z = \mathcal{S}z = \mathcal{B}z = \mathcal{T}z$, and z is unique in $\mathfrak{V}$.

Proof: The proof follows from Remark 3.1 and from the compatible mappings.

Finally, we prove the following theorem for pairs of compatible mappings of type (E).

Theorem 3.3 Let $\mathcal{S}, \mathcal{T}, \mathcal{A}$ and $\mathcal{B}$ are four self mappings of a complete metric space $(\mathfrak{V}, d)$ satisfying (C1) and (C2). Suppose that one of $\mathcal{A}$ and $\mathcal{S}$ is continuous, and one of $\mathcal{B}$ and $\mathcal{T}$ is continuous. Assume that the pairs $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ are compatible of type(E). Then $z = \mathcal{A}z = \mathcal{S}z = \mathcal{B}z = \mathcal{T}z$, and z is unique in $\mathfrak{V}$.

Proof. From the proof of Theorem 2.1,sequence $\{y_n\}$ is a Cauchy sequence in $\mathfrak{V}$, but $(\mathfrak{V}, d)$ is a complete metric space, therefore, $\{y_n\}$converges to a point $z$in $\mathfrak{V}$as $n \to \infty$. Consequently, the subsequences $\{\mathcal{S}x_{2n}\}, \{\mathcal{A}x_{2n}\}, \{\mathcal{T}x_{2n+1}\}$ and $\{\mathcal{B}x_{2n+1}\}$ also converges to the same point z. Now Since the pairs $(\mathcal{A}, \mathcal{S})$ are compatible of type(E) and one of $\mathcal{A}$ and $\mathcal{S}$ is continuous, then by Proposition 2.1, $\mathcal{A}z = \mathcal{S}z$. Since $\mathcal{S}(\mathfrak{V}) \subset \mathcal{B}(\mathfrak{V})$, therefore, there exists a point $q \in \mathfrak{V}$ such that $\mathcal{S}z = \mathcal{B}q$. On putting $u = z$ and $v = q$ in (C2) we get

$$d^3(\mathcal{S}z, \mathcal{T}q) \leq \psi \begin{Bmatrix} d^2(\mathcal{A}z, \mathcal{S}z) d(\mathcal{B}q, \mathcal{T}q), \\ d(\mathcal{A}z, \mathcal{S}z) d^2(\mathcal{B}q, \mathcal{T}q), \\ d(\mathcal{A}z, \mathcal{S}z) d(\mathcal{A}z, \mathcal{T}q) d(\mathcal{B}q, \mathcal{S}z), \\ d(\mathcal{A}z, \mathcal{T}q) d(\mathcal{B}q, \mathcal{S}z) d(\mathcal{B}q, \mathcal{T}q) \end{Bmatrix} - \emptyset \{m(\mathcal{A}z, \mathcal{B}q)\},$$

where $m(\mathcal{A}z, \mathcal{B}q) = max \begin{Bmatrix} d^2(\mathcal{A}z, \mathcal{B}q), \\ d(\mathcal{A}z, \mathcal{S}z) d(\mathcal{B}q, \mathcal{T}q), \\ d(\mathcal{A}z, \mathcal{T}q) d(\mathcal{B}q, \mathcal{S}z), \\ \frac{1}{2}\begin{bmatrix} d(\mathcal{A}z, \mathcal{S}z) d(\mathcal{A}z, \mathcal{T}q) \\ + d(\mathcal{B}q, \mathcal{S}z) d(\mathcal{B}q, \mathcal{T}q) \end{bmatrix} \end{Bmatrix} = 0$

Therefore, we get

$d^3(\mathcal{S}z, \mathcal{T}q) \leq \psi\{0,0,0,0\} - \emptyset(0)$, using property of ψ and $\emptyset$, we have

This implies that $\mathcal{S}z = \mathcal{T}q$. Thus we have $\mathcal{A}z = \mathcal{S}z = \mathcal{T}q = \mathcal{B}q$.

On putting $u = z$ and $v = x_{2n+1}$ in (C2) we get

$$d^3(\mathcal{S}z, \mathcal{T}x_{2n+1})$$

$$\leq \psi \begin{Bmatrix} d^2(\mathcal{A}z, \mathcal{S}z) d(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}), \\ d(\mathcal{A}z, \mathcal{S}z) d^2(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}), \\ d(\mathcal{A}z, \mathcal{S}z) d(\mathcal{A}z, \mathcal{T}x_{2n+1}) d(\mathcal{B}x_{2n+1}, \mathcal{S}z), \\ d(\mathcal{A}z, \mathcal{T}x_{2n+1}) d(\mathcal{B}x_{2n+1}, \mathcal{S}z) d(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}) \end{Bmatrix}$$

$$- \emptyset \{m(\mathcal{A}z, \mathcal{B}x_{2n+1})\},$$

where $m(\mathcal{A}z, \mathcal{B}x_{2n+1}) = max \begin{Bmatrix} d^2(\mathcal{A}z, \mathcal{B}x_{2n+1}), \\ d(\mathcal{A}z, \mathcal{S}z) d(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}), \\ d(\mathcal{A}z, \mathcal{T}x_{2n+1}) d(\mathcal{B}x_{2n+1}, \mathcal{S}z), \\ \frac{1}{2}\begin{bmatrix} d(\mathcal{A}z, \mathcal{S}z) d(\mathcal{A}z, \mathcal{T}x_{2n+1}) \\ + d(\mathcal{B}x_{2n+1}, \mathcal{S}z) d(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}) \end{bmatrix} \end{Bmatrix}$

or $d^3(\mathcal{S}z, z) \le \psi \begin{Bmatrix} d^2(\mathcal{A}z, \mathcal{S}z)d(z, z), \\ d(\mathcal{A}z, \mathcal{S}z)d^2(z, z), \\ d(\mathcal{A}z, \mathcal{S}z)d(\mathcal{A}z, z)d(z, \mathcal{S}z), \\ d(\mathcal{A}z, z)d(z, \mathcal{S}z)d(z, z) \end{Bmatrix} - \emptyset\, \{m(\mathcal{A}z, z)\}$,

where $m(\mathcal{A}z, z) = max \begin{Bmatrix} d^2(\mathcal{A}z, z), \\ d(\mathcal{A}z, \mathcal{S}z)d(z, z), \\ d(\mathcal{A}z, z)d(z, \mathcal{S}z), \\ \frac{1}{2}\begin{bmatrix} d(\mathcal{A}z, \mathcal{S}z)d(\mathcal{A}z, z) \\ +d(z, \mathcal{S}z)d(z, z) \end{bmatrix} \end{Bmatrix} = 0.$

Therefore, we have

$d^3(\mathcal{S}z, z) \le \psi\{0,0,0,0\} - \emptyset(0)$,using property of ψ and $\emptyset$, we have $d^3(\mathcal{S}z, z) = 0$.

This implies that $\mathcal{A}z = \mathcal{S}z = z$.

Now assume that the pair$(\mathcal{B}, \mathcal{T})$are compatible of type (E) and one of $\mathcal{B}$ and $\mathcal{T}$ is continuous. Then we get $\mathcal{B}q = \mathcal{T}q = z$. By Proposition 2.1, we have $\mathcal{B}\mathcal{B}q = \mathcal{B}\mathcal{T}q = \mathcal{T}\mathcal{B}q = \mathcal{T}\mathcal{T}q$, that is $\mathcal{B}z = \mathcal{T}z$. Now we claim that $z = \mathcal{T}z$. On putting $u = x_{2n}$ and $v = z$ in (C2), we have

$$d^3(\mathcal{S}x_{2n}, \mathcal{T}z) \le \psi \begin{Bmatrix} d^2(\mathcal{A}x_{2n}, \mathcal{S}x_{2n})d(\mathcal{B}z, \mathcal{T}z), \\ d(\mathcal{A}x_{2n}, \mathcal{S}x_{2n})d^2(\mathcal{B}z, \mathcal{T}z), \\ d(\mathcal{A}x_{2n}, \mathcal{S}x_{2n})d(\mathcal{A}x_{2n}, \mathcal{T}z)d(\mathcal{B}z, \mathcal{S}x_{2n}), \\ d(\mathcal{A}x_{2n}, \mathcal{T}z)d(\mathcal{B}z, \mathcal{S}x_{2n})d(\mathcal{B}z, \mathcal{T}z) \end{Bmatrix} - \emptyset\, \{m(\mathcal{A}x_{2n}, \mathcal{B}z)\},$$

where $m(\mathcal{A}x_{2n}, \mathcal{B}z) = max \begin{Bmatrix} d^2(\mathcal{A}x_{2n}, \mathcal{B}z), \\ d(\mathcal{A}x_{2n}, \mathcal{S}x_{2n})d(\mathcal{B}z, \mathcal{T}z), \\ d(\mathcal{A}x_{2n}, \mathcal{T}z)d(\mathcal{B}z, \mathcal{S}x_{2n}), \\ \frac{1}{2}\begin{bmatrix} d(\mathcal{A}x_{2n}, \mathcal{S}x_{2n})d(\mathcal{A}x_{2n}, \mathcal{T}z) \\ +d(\mathcal{B}z, \mathcal{S}x_{2n})d(\mathcal{B}z, \mathcal{T}z) \end{bmatrix} \end{Bmatrix} = d^2(z, \mathcal{T}z)$

Proceeding limit as $n \to \infty$,weget

$d^3(z, \mathcal{T}z) \le \psi\{0,0,0,0\} - \emptyset\{d^2(z, \mathcal{T}z)\}$.This implies that $z = \mathcal{T}z$.

Uniqueness follows easily. Then $z = \mathcal{A}z = \mathcal{S}z = \mathcal{B}z = \mathcal{T}z$, and z is unique in $\mathfrak{V}$.

At the last, we prove a common fixed point theorem for pairs of intimate mappings. In fact intimate mappings are generalizations of compatible mappings of type (A).

Theorem 3.4 Let $\mathcal{S}, \mathcal{T}, \mathcal{A}$ and $\mathcal{B}$ are four self mappings of a complete metric space $(\mathfrak{V}, d)$ satisfying (C1) and (C2) and the following conditions:

(3.4) the pair $(\mathcal{A},\mathcal{S})$ is $\mathcal{A}$ -intimate and pair $(\mathcal{B},\mathcal{T}$ is $\mathcal{B}$ -intimate;

(3.5) $\mathcal{A}(\mathfrak{V})$ is a complete subspace of $\mathfrak{V}$.

Then $q = \mathcal{A}q = \mathcal{S}q = \mathcal{B}q = \mathcal{T}q$, and q is unique in $\mathfrak{V}$.

Proof Let $x_0 \in \mathfrak{V}$ be an arbitrary point. From (C1) we can find x_1 such that $\mathcal{S}(x_0) = \mathcal{B}(x_1) = y_0$ for this x_1 one can find $x_2 \in \mathfrak{V}$ such that $\mathcal{T}(x_1) = \mathcal{A}(x_2) = y_1$.Continuing in this way, one can construct a sequence $\{x_n\}$ such that

$$y_{2n} = \mathcal{S}(x_{2n}) = \mathcal{B}(x_{2n+1}),$$

$y_{2n+1} = \mathcal{T}(x_{2n+1}) = \mathcal{A}(x_{2n+2}),$ for each $n \geq 0$.

From the proof of Theorem 2.1, the sequence $\{y_n\}$ is a Cauchy sequence in $\mathfrak{V}$. Since $\mathcal{A}(\mathfrak{V})$ is complete, $\exists$ a point $q \in \mathcal{A}\mathfrak{V}$ such that $y_{2n+1} = \mathcal{T}(x_{2n+1}) = \mathcal{A}(x_{2n+2}) \to q$ as $n \to \infty$.

Consequently, we find $p \in \mathfrak{V}$ such that $\mathcal{A}p = q$. Since $\{y_n\}$ is a Cauchy sequence containing a convergent subsequence $\{y_{2n+1}\}$, therefore the sequence $\{y_n\}$ also converges, which implies the convergence of $\{y_{2n}\}$, being a subsequence of the convergent sequence $\{y_n\}$. Hence $\{\mathcal{S}(x_{2n})\}, \{\mathcal{B}(x_{2n+1})\}, \{\mathcal{T}(x_{2n+1})\}, \{\mathcal{A}(x_{2n+2})\}$ converges to q.

Now we claim that $\mathcal{S}p = q$. On putting $u = p$ and $v = x_{2n+1}$ in (C2) we get

$$d^3(\mathcal{S}p, \mathcal{T}x_{2n+1})$$

$$\leq \psi \left\{ \begin{array}{c} d^2(\mathcal{A}p, \mathcal{S}p) d(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}), \\ d(\mathcal{A}p, \mathcal{S}p) d^2(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}), \\ d(\mathcal{A}p, \mathcal{S}p) d(\mathcal{A}p, \mathcal{T}x_{2n+1}) d(\mathcal{B}x_{2n+1}, \mathcal{S}p), \\ d(\mathcal{A}p, \mathcal{T}x_{2n+1}) d(\mathcal{B}x_{2n+1}, \mathcal{S}p) d(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}) \end{array} \right\}$$

$$- \emptyset \{m(\mathcal{A}p, \mathcal{B}x_{2n+1})\},$$

Where $m(\mathcal{A}p, \mathcal{B}x_{2n+1}) = max \left\{ \begin{array}{c} d^2(\mathcal{A}p, \mathcal{B}x_{2n+1}), \\ d(\mathcal{A}p, \mathcal{S}p) d(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}), \\ d(\mathcal{A}p, Tx_{2n+1}) d(\mathcal{B}x_{2n+1}, \mathcal{S}p), \\ \frac{1}{2}\left[\begin{array}{c} d(\mathcal{A}p, \mathcal{S}p) d(\mathcal{A}p, \mathcal{T}x_{2n+1}) \\ + d(\mathcal{B}x_{2n+1}, \mathcal{S}p) d(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}) \end{array} \right] \end{array} \right\}$

Proceeding limit as $n \to \infty$,weget

$d^3(\mathcal{S}p, q) \leq \psi\{0,0,0,0\} - \emptyset(d^2(\mathcal{A}p, q)),$

using property of ψ and $\emptyset$, we have $d^3(\mathcal{S}p, q) = 0$, which implies $\mathcal{S}p = q$.

Therefore, $\mathcal{A}p = \mathcal{S}p = q$.

Since $q = \mathcal{S}p \in \mathcal{S}(\mathfrak{V}) \subset \mathcal{B}(\mathfrak{V}), \exists$ a point w in $\mathfrak{V}$ such that $\mathcal{B}w = q$.

Next, we claim that $q = \mathcal{T}w$. On putting $u = p$ and $v = w$ in (C2) we get

$$d^3(\mathcal{S}p, \mathcal{T}w) \le \psi \left\{ \begin{matrix} d^2(\mathcal{A}p, \mathcal{S}p)d(\mathcal{B}w, \mathcal{T}w), \\ d(\mathcal{A}p, \mathcal{S}p)d^2(\mathcal{B}w, \mathcal{T}w), \\ d(\mathcal{A}p, \mathcal{S}p)d(\mathcal{A}p, \mathcal{T}w)d(\mathcal{B}w, \mathcal{S}p), \\ d(\mathcal{A}p, \mathcal{T}w)d(\mathcal{B}w, \mathcal{S}p)d(\mathcal{B}w, \mathcal{T}w) \end{matrix} \right\} - \emptyset\,\{m(\mathcal{A}p, \mathcal{B}w)\},$$

where $m(\mathcal{A}p, \mathcal{B}w) = max \left\{ \begin{matrix} d^2(\mathcal{A}p, \mathcal{B}w), \\ d(\mathcal{A}p, \mathcal{S}p)d(\mathcal{B}w, \mathcal{T}w), \\ d(\mathcal{A}p, \mathcal{T}w)d(\mathcal{B}w, \mathcal{S}p), \\ \frac{1}{2}\left[\begin{matrix} d(\mathcal{A}p, \mathcal{S}p)d(\mathcal{A}p, \mathcal{T}w) \\ +d(\mathcal{B}w, \mathcal{S}p)d(\mathcal{B}w, \mathcal{T}w) \end{matrix} \right] \end{matrix} \right\} = 0.$

On simplification, we have

$$d^3(q, \mathcal{T}w) \le \psi\{0,0,0,0\} - \emptyset(0)$$

Thus we get $d^3(q, \mathcal{T}w) = 0$, which implies that$q = \mathcal{T}w$.

Hence $\mathcal{B}w = \mathcal{T}w = q$.

Since $\mathcal{A}p = \mathcal{S}p = q$ and the pair $(\mathcal{A}, \mathcal{S})$ is $\mathcal{A}$ -intimate, by Proposition 3.3, we have

$$d(\mathcal{A}q, q) \le d(\mathcal{S}q, q).$$

Next, we claim that $q = \mathcal{S}q$. On putting $u = q$ and $v = w$ in (C2) we get

$$d^3(\mathcal{S}q, \mathcal{T}w) \le \psi \left\{ \begin{matrix} d^2(\mathcal{A}q, \mathcal{S}q)d(\mathcal{B}w, \mathcal{T}w), \\ d(\mathcal{A}q, \mathcal{S}q)d^2(\mathcal{B}w, \mathcal{T}w), \\ d(\mathcal{A}q, \mathcal{S}q)d(\mathcal{A}q, \mathcal{T}w)d(\mathcal{B}w, \mathcal{S}q), \\ d(\mathcal{A}q, \mathcal{T}w)d(\mathcal{B}w, \mathcal{S}q)d(\mathcal{B}w, \mathcal{T}w) \end{matrix} \right\} - \emptyset\,\{m(\mathcal{A}q, \mathcal{B}w)\},$$

Where $m(\mathcal{A}q, \mathcal{B}w) = max \left\{ \begin{matrix} d^2(\mathcal{A}q, \mathcal{B}w), d(\mathcal{A}q, \mathcal{S}q)d(\mathcal{B}w, \mathcal{T}w), \\ d(\mathcal{A}q, \mathcal{T}w)d(\mathcal{B}w, \mathcal{S}q), \\ \frac{1}{2}\left[\begin{matrix} d(\mathcal{A}q, \mathcal{S}q)d(\mathcal{A}q, \mathcal{T}w) \\ +d(\mathcal{B}w, \mathcal{S}q)d(\mathcal{B}w, \mathcal{T}w) \end{matrix} \right] \end{matrix} \right\} = 0.$

Therefore,

$d^3(\mathcal{S}q, q) \le \psi\{0,0,0,0\} - \emptyset(0).$

Thus we get $d^3(\mathcal{S}q, q) = 0$,which further implies that $\mathcal{S}q = q$.

Hence $\mathcal{S}q = \mathcal{A}q = q$.

Similarly, we get $\mathcal{B}q = \mathcal{T}q = q$.

The uniqueness follows easily. Hence $q = \mathcal{A}q = \mathcal{S}q = \mathcal{B}q = \mathcal{T}q$, and q is unique in $\mathfrak{V}$.

4. APPLICATION

In 2002 Branciari [4] obtained a fixed point theorem for a single mapping satisfying an analogue of a Banach contraction principle for integral type inequality.

Theorem 4.1 Let $(\mathfrak{V}, d)$ be a complete metric space and $f : \mathfrak{V} \to \mathfrak{V}$ is a mapping such that, for each $x, y \in \mathfrak{V}$,

$$\int_0^{d(x,y)} \varphi(t)\, dt \leq c \int_0^{d(x,y)} \varphi(t)\, dt$$

$c \in [0, 1)$, where $\varphi : R^+ \to R^+$ is a "Lebesgue-integrable function" which is summable, nonnegative, and such that, for each $\in > 0$, $\int_0^{\in} \varphi(t) dt > 0$. Then f has a unique fixed point $z \in \mathfrak{V}$ such that, for each $x \in \mathfrak{V}$, $\lim_{n\to\infty} f^n = z$.

Now we prove the following theorem as an application of theorem 3.1.

Theorem 4.2 Let $\mathcal{S}, \mathcal{T}, \mathcal{A}$ and $\mathcal{B}$ be four self-mappings of a complete metric space $(\mathfrak{V}, d)$ satisfying the conditions (C1), and the following conditions:

(C3)

$$\int_0^{d^3(\mathcal{S}x,\mathcal{T}y)} \varphi(t)\, dt \leq \int_0^{M(x,y)} \varphi(t)\, dt$$

$$M(u,v) = \psi \begin{Bmatrix} d^2(\mathcal{A}u,\mathcal{S}u)d(\mathcal{B}v,\mathcal{T}v), \\ d(\mathcal{A}u,\mathcal{S}u)d^2(\mathcal{B}v,\mathcal{T}v), \\ d(\mathcal{A}u,\mathcal{S}u)d(\mathcal{A}u,\mathcal{T}v)d(\mathcal{B}v,\mathcal{S}u), \\ d(\mathcal{A}u,\mathcal{T}v)d(\mathcal{B}v,\mathcal{S}u)d(\mathcal{B}v,\mathcal{T}v) \end{Bmatrix} - \emptyset\,\{m(\mathcal{A}u,\mathcal{B}v)\},$$

where $m(\mathcal{A}u,\mathcal{B}v) = max \begin{Bmatrix} d^2(\mathcal{A}u,\mathcal{B}v), d(\mathcal{A}u,\mathcal{S}u)d(\mathcal{B}v,\mathcal{T}v), \\ d(\mathcal{A}u,\mathcal{T}v)d(\mathcal{B}v,\mathcal{S}u), \\ \frac{1}{2}\begin{bmatrix} d(\mathcal{A}u,\mathcal{S}u)d(\mathcal{A}u,\mathcal{T}v) \\ +d(\mathcal{B}v,\mathcal{S}u)d(\mathcal{B}v,\mathcal{T}v) \end{bmatrix} \end{Bmatrix}$

for all $u, v \in \mathfrak{V}$, where ψ: $[0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function with $\psi(t) < t$ for each $t > 0$ and $\emptyset$: $[0, \infty) \to [0, \infty)$ is a continuous function with $\emptyset\,(t) = 0 \Leftrightarrow t = 0$ and $\emptyset(t) > 0$ for each $t > 0$. Further, where $\varphi : R^+ \to R^+$ is a "Lebesgue-integrable over R^+function" which is summable on each compact subset of R^+, non-negative, and such that for each $\in > 0$, $\int_0^{\in} \varphi(t)dt > 0$. Moreover, assume that the pairs the pairs $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ are compatible of type (K). Then $z = \mathcal{A}z = \mathcal{S}z = \mathcal{B}z = \mathcal{T}z$, and z is unique in $\mathfrak{V}$.

Proof. The proof of the theorem follows on the same lines of the proof of the theorem 3.1. On setting φ (t) = 1, we get theorem 3.1.

Remark 4.1 Every contractive condition of integral type automatically includes a corresponding contractive condition not involving integrals, by setting φ (t) = 1 .

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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