ON THE MEAN VALUES OF AN ENTIRE FUNCTION REPRESENTED BY A DIRICHLET SERIES

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ABSTRACT
In this paper, we obtain some results for the mean value of an entire Dirichlet series.

THEOREM 1. (*i*) For $0<k<\infty, \delta>1$

$$\frac{\rho_*}{\lambda_*} \le \lim_{\sigma\to\infty} {\sup \atop \inf} \frac{\log\log N_{\delta,k}(\sigma)}{\sigma} \le \frac{\rho}{\lambda} \tag{a}$$

Under the additional condition on $\{\lambda_n\}$,

$$0 \le \lim_{n\to\infty} \sup \frac{\log n}{\lambda_n} = D < \infty, \tag{A}$$

(a) Becomes

$$\lim_{\sigma\to\infty} {\sup \atop \inf} \frac{\log\log N_{\delta,k}(\sigma)}{\sigma} = \frac{\rho}{\lambda} = \frac{\hat{\rho}}{\hat{\lambda}} \tag{b}$$

(*ii*) For $0<k<\infty, \delta>0$

$$\lim_{\sigma\to\infty} {\sup \atop \inf} \frac{\log\log N_{\delta,k}(\sigma)}{\sigma} \le \frac{\rho}{\lambda} \tag{c}$$

In fact for the truth of 'lim sup' part of (b) the following condition on $\{\lambda_n\}$ is sufficient.

$$\lim_{n\to\infty} \frac{\log n}{\lambda_n \log \lambda_n} = 0. \tag{A'}$$

THEOREM 2. (*i*) For $\delta>0$, $0<k<\infty$,

$$\lim_{\sigma\to\infty} {\sup \atop \inf} \frac{\log N_{\delta,k}(\sigma)}{e^{\rho\sigma}} \le \frac{T}{t}, (0<\rho<\infty). \tag{d}$$

(*ii*) For $\delta \ge 1$, $0<k<\infty$ and under the additional condition (A)

$$\frac{T_*}{t_*} < \lim_{\sigma\to\infty} {\sup \atop \inf} \frac{\log N_{\delta,k}(\sigma)}{e^{\rho\sigma}} \le \frac{T}{t} < \frac{T_*}{t_*} e^{\rho D} \tag{e}$$

In particular case, if D = 0,

$$\lim_{\sigma\to\infty} {\sup \atop \inf} \log \frac{N_{\delta,k}(\sigma)}{e^{\rho\sigma}} = \frac{T_*}{t_*} = \frac{T}{t} \tag{f}$$

KEYWORDS: - Generalized order ρ
Generalized lower order λ

INTRODUCTION: In the usual notation,

$$f(s)=\sum_{1}^{\infty} a_n e^{s\lambda_n}, (s=\sigma+it), 0<\lambda_n<\lambda_{n+1} \quad (n\ge 1) \lim_{n\to\infty} \lambda_n = \infty,$$

Is an entire function in the sense that the Dirichlet series representing it, is absolutely convergent for all finite s and possesses two generally different pairs of orders:

$$\lim_{\sigma\to\infty} {\sup \atop \inf} \frac{\log\log M(\sigma)}{\sigma} = \frac{\rho}{\lambda};$$

is an absolutely and uniformly convergent function of t and hence ([2], p.6) a function of t which i uniformly almost periodic (briefly $u.a.p.$) $|f(\sigma+it)|^{\delta}, \delta>0$ is also a function of t which is $u.a.p.$, as shown b familiar considerations ($e.g.$ as in [2] p.3) involving the following well known inequalities for $a>0$, $b>0$.

$$(a+b)^{\delta} \le a^{\delta}+b^{\delta} \text{ if } 0<\delta<1, a^{\delta}-b^{\delta} \le \delta_a^{\delta-1}(a-b), \text{ if } \delta \ge 1, a \ge b.$$

By the result ([2], p. 12) the mean value of $|f(\sigma+it)|^{\delta}, \delta>0$, defined by $A_{\delta}(\sigma)$ exists.

For $\delta>0$ it is obvious that

$$I_{\delta}(\sigma) \le M(\sigma).$$

This, with (1.2) will give us

$$N_{\delta,k}(\sigma) \le \frac{M(\sigma)}{k}. \tag{2.4}$$

From which it follows that

$$\lim_{\sigma\to\infty} {}^{\sup}_{\inf} \frac{\log\log N_{\delta,k}(\sigma)}{\sigma} \le {}^{\rho}_{\lambda},\ 0<k<\infty,\ \delta>0. \tag{2.5}$$

This formula gives us

$$\mu(\sigma) \le \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(\sigma+it)|dt = I_1(\sigma)$$

If $\delta>1$, we also get by Holder integral inequality $\mu(\sigma) \le \lim_{T\to\infty}\left[\frac{1}{2T}\int_{-T}^{T}|f(\sigma+it)|^{\delta}dt\right]^{\frac{1}{\delta}}\left[\frac{1}{2T}\int_{-T}^{T}dt\right]^{\frac{1}{\delta'}}$

where $\frac{1}{\delta}+\frac{1}{\delta'}=1$. Hence

$$\mu(\sigma) \le I_{\delta}(\sigma) \quad for\ \delta \ge 1.$$

From (1.2), we have for $h>0$,

$$N_{\delta,k}(\sigma+h) \ge \frac{\mu(\sigma)}{k}(1-e^{-kh}) \tag{2.6}$$

This leads to

$$\frac{\log\log N_{\delta,k}(\sigma+h)}{(\sigma+h)} \ge \frac{\log\log\mu(\sigma)}{(\sigma+h)}+o(1)$$

Proceedings to limits, we get

$$\lim_{\sigma\to\infty} {}^{\sup}_{\inf} \frac{\log\log N_{\delta,k}(\sigma)}{\sigma} \ge {}^{\rho_*}_{\lambda_*} \tag{2.7}$$

Combining (2.5) and (2.7), we get

$${}^{\rho_*}_{\lambda_*} < \lim_{\sigma\to\infty} {}^{\sup}_{\inf} \frac{\log\log N_{\delta,k}(\sigma)}{\sigma} \le {}^{\rho}_{\lambda}$$

To prove (2.2), we use the known result ([10], p. 68) that, under the condition (A),

$$M(\sigma) < K\,\mu(\sigma+D+\varepsilon)$$

where ε is an arbitrary small positive number. K is a constant depending on D and ε. This give $\rho \le \rho_*$ and $\lambda \le \lambda_*$ but $\rho_* < \rho$ and $\lambda_* < \lambda$ always. Thus, (2.2) proved.

It is known that under the condition (A')

$$\rho = \lim_{n\to\infty}\sup\frac{\lambda_n\log\lambda_n}{\log|a_n|^{-1}}, [1].$$

Further, from the result of Reddy [8] we conclude that

$$\rho_* = \lim_{n\to\infty}\sup\frac{\lambda_n\log\lambda}{\log|a_n|^{-1}}.$$

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$$\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\log\mu(\sigma)}{\sigma}=\frac{\rho_*}{\lambda_*};$$

Where $0\le\lambda,\rho\le\infty, 0\le\lambda_*,\rho_*\le\infty, and\ M(\sigma),\mu(\sigma)$ their usual meanings, viz.

$$M(\sigma)=\underset{-\infty<t<\infty}{l.u.b.}\left|f(\sigma+it)\right|,\quad \mu(\sigma)=\max_{n\ge1}\left|a_n e^{(\sigma+it)\lambda_n}\right|$$

The type T, t associated with ρ and type T$_*$, t_* associate with ρ_* are defined in the usual way as follo

$$\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log M(\sigma)}{e^{\rho\sigma}}=\frac{T}{t},\quad (0<\rho<\infty)$$

$$\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\mu(\sigma)}{e^{\rho_*\sigma}}=\frac{T_*}{t_*},\quad (0<\rho_*<\infty).$$

The mean values of t (s) are defined as follows:

$$\{I_\delta(\sigma)\}^\delta=A_\delta(\sigma)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left|f(\sigma+it)\right|^\delta dt,\quad 0<\delta<\infty, \tag{1.1}$$

$$N_{\delta,k}(\sigma)=\frac{1}{e^{k\sigma}}\int_{-\infty}^{\sigma}I_\delta(x)e^{kx}dx$$

$$=\lim_{T\to\infty}\frac{1}{2Te^{k\sigma}}\int_{-\infty}^{\sigma}\int_{-T}^{T}\left|f(x+it)\right|^\delta e^{kx}dx\,dt,\quad \begin{matrix}0<\delta<\infty\\ 0<k<\infty\end{matrix} \tag{1.2}$$

Clearly $\rho_*\le\rho$ and $\lambda_*\le\lambda$. There are entire Dirichlet series for which $\rho_*<\rho,\lambda_*<\lambda$ (sec[9], *Satz* 4 So, we have generally to distinguish between the two orders of an entire Dirichlet series and its typ associated with these orders.

THEOREM 1. *(i) For* $0<k<\infty,\ \delta\ge1$

$$\frac{\rho_*}{\lambda_*}\le\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\log N_{\delta,k}(\sigma)}{\sigma}\le\frac{\rho}{\lambda} \tag{2.1}$$

Under the additional condition on $\{\lambda_n\}$,

$$0\le\lim_{n\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log n}{\lambda_n}=D<\infty, \tag{A}$$

(2.1) becomes

$$\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\log N_{\delta,k}(\sigma)}{\sigma}=\frac{\rho}{\lambda}=\frac{\rho_*}{\lambda_*} \tag{2.2}$$

(ii) For $0<k<\infty,\delta>0$

$$\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\log N_{\delta,k}(\sigma)}{\sigma}\le\frac{\rho}{\lambda} \tag{2.3}$$

In fact for the truth of 'lim sup' part of (2.2) the following condition on $\{\lambda_n\}$ is sufficient.

$$\lim_{\sigma\to\infty}\frac{\log n}{\lambda_n\log\lambda_n}=0. \tag{A'}$$

Proof. For fixed σ,

$$f(\sigma+it)=\sum_{1}^{\infty}(a_n e^{\lambda_n\sigma})e^{i\lambda_n t},\quad (-\infty<t<\infty)$$

We have completed the proof of the theorem

THEOREM 2. (*i*) *For* $\delta > 0, 0 < k < \infty$,

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log N_{\delta,k}(\sigma)}{e^{\rho\sigma}} \le {}^{T}_{t}, (0 < \rho < \infty). \tag{3.1}$$

(*ii*) *For* $\delta \ge 1, 0 < k < \infty$ and under the condition (A)

$${}^{T_*}_{t_*} \le \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log N_{\delta,k}(\sigma)}{e^{\rho\sigma}} \le {}^{T}_{t} \le {}^{T_*}_{t_*} {}^{e^{\rho D}}_{e^{\rho D}} \tag{3.2}$$

In particular case, if D = 0,

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \log \frac{N_{\delta,k}(\sigma)}{e^{\rho\sigma}} = {}^{T_*}_{t_*} = {}^{T}_{t} \tag{3.3}$$

Proof. From (2.4), we get

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log N_{\delta,k}(\sigma)}{e^{\rho\sigma}} \le \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(\sigma)}{e^{\rho\sigma}}.$$

From which (3.1) follows.

To prove (3.2), we use (2.4), (2.6) and the known result $M(\sigma) < K\,\mu(\sigma + D + \varepsilon)$[10], where ε is an arbitrary small positive number and K is constant depending on D and ε. We have

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \mu(\sigma)}{e^{\rho\sigma}} \le \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(\sigma)}{e^{\rho\sigma}} \le \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \mu(\sigma + D + \varepsilon)}{e^{\rho\sigma}}$$

And

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \mu(\sigma)}{e^{\rho\sigma}} \le \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log N_{\delta,k}(\sigma)}{e^{\rho\sigma}} \le \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(\sigma)}{e^{\rho\sigma}}.$$

Combining these two, we get desired conclusion (3.2). The particular case (3.3) is obvious.

Conclusion: . Our theorem includes the results of Jain [5], which in turn includes the theorem of Juneja [6] and also a theorem of Gupta [3]. The method of proofs of our results is different from that of Jain. Jain has used in his proof the following result of Kamthan [7]

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