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APPLICATION TO MEDICAL SCIENCE USING G-METRIC SPACES

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Abstract

In this paper, we have extended the concept of contraction by introducing D-Contraction defined on a family of $\Im$ of bounded function . Also, a new notion of fixed function has been extended G-metric space. Some fixed theorems with illustrate examples have also been given to verify the effectiveness of our result. In 2019, Vishal et. al [3] presented some fixed function theorems using the notion of fixed function and D-Contraction in complete metric space. Following this concept, a sequence is constructed in G-metric space with three different dose function which converges to a fixed function. An application based on the best approximation of treatment plan for tumor patients by splitting DDC matrix into three components with three dose functions.

Keywords: Fixed function, G-metric space, D-Contraction, $\alpha - \psi$ contractive mapping.

1 Introduction

Banach contraction principle is one of the most fruitful and applicable theorems in the classical functional analysis. This principle ensures that the application of a continuous self mapping on two points of a complete metric space contracts the distance between those two points. Stefan Banach [12] gave the concept of "*Contraction* to prove the existence and uniqueness of a fixed point. Banach Contraction says that, *"A contraction self mapping defined on a complete metric space possesses a unique fixed point which can be obtained as the limit of an iteration scheme constructed by applying repeated images of the mapping (starting from an arbitrary point of space)"*. For more details, references $[1, 6, 9-11]$ can be cited

After that, many authors named Kannan [11], Chatterjea [13] and Ciric [7] gave the extension of this result by applying different contraction conditions.

By now, there exists a literature on all these generalizations in different spaces which are applicable to the numerous fields. This paper deals with a unique approach in the field

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of contraction mappings introduced with a family of bounded functions. In section 3, main results are presented with some illustrative examples while section 4 deals with an application to the medical science.
In order to prove our main results, we have taken some basic concepts, definitions and results from the literature.

2 Preliminaries

Mustafa and Sims [15] introduced a new class of generalized metric space called G-metric space in 2005 which is a generalization of metric sapce (M, d).

Definition 2.1. [15] Let M be a non empty set, and $G : M \times M \times M \to R^+$ be a function satisfying the following properties:

1. $G(l, m, n) = 0$ if $l = m = n$,
2. $0 < G(l, l, m)$, for all $l, m \in M$, with $l \neq m$,
3. $G(l, l, m) \leqslant G(l, m, n)$, for all $l, m, n \in M$, with $n \neq m$,
4. $G(l, m, n) = G(l, n, m) = G(m, n, l) = ...$(symmetry in all three variables),
5. $G(l, m, n) \leq G(l, a, a) + G(a, m, n)$, for all $l, m, n, a \in M$, (rectangular inequality).

Then the function G is called a generalized metric, or, more specifically a G-metric on M, and the pair (M, G) is called a *G-metric space.*

Definition 2.2. [15] For a G-metric space (M, G), a mapping $T : M \to M$ is called a contraction mapping on M if for any real number λ with $0 \leq \lambda < 1$, the following inequality holds:

$$G(Tl, Tm, Tn) \leq \lambda G(l, m, n) \text{ for all } l, m, n \in M.$$

Remark 2.3. It can be easily seen that the geographical distance between the images of any three points of a given set is contracting by a uniform factor $\lambda < 1$.

Example 2.4. [15] Let $M = R^3$ be a set equipped with standard G-metric $G(i.e.G(l, m, n) = |l_1 - l_2| + |l_2 - l_3| + |m_1 - m_2| + |m_2 - m_3| + |n_1 - n_2| + |n_2 - n_3| forall l, m, n \in M)$ and $T : R^3 \to R^3$ be the mapping defined as $Tl = \frac{5}{8}l$ for all $l \in R^3$. Then T is a contraction on M as $G(l, m, n) = \frac{5}{8}\{|l_1 - m_1 - n_1| + |l_2 - m_2 - n_2| + |l_3 - m_3 - n_3|\} = \frac{5}{8}G(l, m, n)$.

Theorem 2.5. [15] *Let (M, G) be a complete G-metric space and T be the contraction mapping defined on M. Then T possesses a unique fixed point l in M i.e. $Tl = l$.*

Theorem 2.6. [16] *Let (M, G) be a complete G-metric space and T be the self mapping defined on M which satisfy the condition*

$$G(Tl, Tm, Tn) \leq \alpha G(l, Tl, Tl) + \beta G(m, Tm, Tm) + \gamma G(n, Tn, Tn) + \delta G(l, m, n)$$

for all $l, m, n \in M$ and $\alpha, \beta, \gamma, \delta$ non negative with $\alpha + \beta + \gamma + \delta < 1$. Then T admits a unique fixed point in M.

Definition 2.7. [8] Let Ψ be the family of all functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying the following properties:

1. $\Sigma_{p=1}^{+\infty} \psi^p(t) < +\infty$ for every $t > 0$, where ψ^p is the pth iterate of ψ;

2. ψ is nondecreasing.

Lemma 2.8. [2] *If $\psi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing function, then for each $t > 0, \lim_{p \to +\infty} \psi^p(t) = 0$ implies $\psi(t) < t$.*

Lemma 2.9. [2] *If $\psi \in \Psi$, then the function ψ is continuous at 0.*

Definition 2.10. [8] Let (M, G) be a G-metric space and T be a given self mapping defined on M. The mapping T is said to be an $\alpha - \psi$ contractive mapping if there exists two functions $\alpha : M \times M \times M \to [0, +\infty$ and $\psi \in \Psi$ satisfying

$$\alpha(l, m, n) G(Tl, Tm, Tn) \leq \psi(G(l, m, n)) \text{ for all } l, m, n \in M.$$

Definition 2.11. [8] Let $T : M \to M$ and $\alpha : M \times M \times M \to [0, +\infty$. The mapping T is known as α-admissible mapping if

$$\alpha(l, m, n) \geq 1 \Rightarrow \alpha(Tl, Tm, Tn) \geq 1 \text{ for every } l, m, n \in M.$$

3 Main Results

This section presents some fixed function theorems using the notions of fixed function and D-Contraction.

Definition 3.1. Let D be any self mappings defined on a family of functions $\Im$, then $f \in \Im$ is said to be fixed function of D if $Df = f$.

Definition 3.2. Let $(M, \hat{G})$ be a complete G-metric space and let $\Im$ be the collection of all bounded functions defined on M. Let D be any self mapping on $\Im$. Then the given mapping is called D-co0ntraction mapping on $\Im$, if for any real number $\lambda \in [0, 1)$, we have

$$G^*(Df, Dg, Dh) \leq \lambda G^*(f, g, h) \text{ for all } f, g, h \in \Im$$

where

$$G^*(f, g, h) = \sup\{\hat{G}(f(l), g(m), h(n)) | l, m, n \in M\} = \sup\{|f(l) - g(m)| + |g(m) - h(n)| + |f(l) - h(n)| | l, m, n \in M\} \tag{3.1}$$

Theorem 3.3. *Let (M, G) be a complete G-metric space with metric G defined as $G(l, m, n) = max\{|l - m|, |m - n|, |n - l|\}$ for all $l, m, n \in G$. Let $\Im$ be a collection of all bounded functions f defined on M with G^*.*
Also, let D-contraction mapping defined on $\Im$. Then there exists a unique fixed function $f \in \Im$ i.e. there exists some $f \in \Im$ such that $Df = f$.

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Proof. Let f, g be any two functions from the family $\Im$. Since D is the D-contraction mapping on $\Im$, therefore, there exists a real number $\lambda \in [0, 1)$ such that

$$G^*(Df, Dg, Dg) \leq \lambda G^*(f, g, g) \text{ for all } f, g \in \Im$$

where

$$G^*(f, g, g) = sup\{\hat{G}(f(l), g(m), g(m))\} \text{ such that } l, m \in M.$$

This further implies that

$$G^*(D^2 f, D^2 g, D^2 g) \leq \lambda G^*(Df, Dg, Dg)$$
$$\leq \lambda^2 G^*(f, g, g) for all f, g \in \Im.$$

Continuing in the same manner, we get

$$G^*(D^p f, D^p g, D^p g) \leq \lambda^p G^*(f, g, g) for all f, g \in \Im. \tag{3.2}$$

Step I: *We will show that* $\{f_n\}_{p\in N}$ *is a cauchy sequence.*
Let f_0 be any function in $\Im$. Let us define the sequence $\{f_p\}_{(p\in N)}$ by setting

$$f_1 = D(f_0),$$
$$f_2 = D(f_1) = D^2(f_0),$$
$$...$$
$$...$$
$$f_p = D(f_{p-1}) = D^2(f_{p-2}) = ... = D^n(f_0).$$

Let $a, b \in N$ be some positive integers with $a > b$. Let $a = b + x$ where x is a +ve integer greater than equal to 1.

$$G^*(f_b, f_a, f_a) = G^*(f_b, f_{b+x}, f_{b+x})$$

$$\leq G^*(f_b, f_{b+1}, f_{b+1}) + G^*(f_{b+1}, f_{b+2}, f_{b+2}) + ... + G^*(f_{b+x-1}, f_{b+x}, f_{b+x})$$

$$= G^*(D^b f_0, D^b f_1, D^b f_1) + G^*(D^{b+1} f_0, D^{b+1} f_1, D^{b+1} f_1) + ... +$$
$$G^*(D^{b+x-1} f_0, D^{b+x-1} f_1, D^{b+x-1} f_1)$$

$$\leq \lambda^b G^*(f_0, f_1, f_1) + \lambda^{b+1} G^*(f_0, f_1, f_1) + ... + \lambda^{q+x-1} G^*(f_0, f_1, f_1) \text{ (using (3.2))}$$

$$= \lambda^b G^*(f_0, f_1, f_1).[1 + \lambda + \lambda^2 + ... + \lambda^{x-1}]$$

$$\leq \frac{\lambda^b}{1-\lambda} G^*(f_0, f_1, f_1) where \lambda < 1.$$

Since $\Im$ is a family of bounded functions, therefore

$$G^*(f_b, f_a, f_a) \to 0 as a, b \to \infty.$$

Hence $\{f_p\}_{(p\in N)}$ is a cauchy sequence in $\Im$.

Step II:*Existence of fixed function.* As $\Im$ is the family of bounded functions defined on complete G-metric space (M,G), therefore, $(D,\hat{G})$, therefore $(\Im, G^*)$ is a complete G-metric space and thus the sequence $\{f_p\}_{(p\in N)}$ is convergent in $\Im$.
Let $f\in\Im$ be the limit of $\{f_p\}_{(p\in N)} i.e. \lim_{p\to\infty} f_p = f$.
By the continuity of D, we get

$$\lim_{n\to\infty} Df_p = Df.$$

Also,

$$Df_p = f_{n+1} \to f asp \leftarrow \infty.$$

Thus, uniqueness of limit implies that $Df = f$. This shows that f is a fixed function of D.

Step III:*Uniqueness of fixed function.* Let g be another fixed function of D i.e. $Dg = g$ and $f \nsim g$.

$$\begin{aligned} 0 &\leq G^*(Df, Dg, Dg) \\ &\leq \lambda G^*(f, g, g) \\ &< G^*(f, g, g). \end{aligned}$$

Thus we arrive at a contradiction. Hence, f is a unique fixed function of D. □

Example 3.4. Let $M = R$(set of real numbers) and $\hat{G}$ be the G-metric space defined on R. Clearly, $(M,\hat{G})$ is a complete G-metric space. Let D be the family of bounded functions defined on M and G^* be the G-metric on D defined as

$$G^*(f,g,g) = sup\{\hat{G}(f(l), g(m), g(m))|l, m \in M\} = sup\{|f(l) - g(m)||l, m \in M\}.$$

It can be easily seen that (D, G^*) is a complete G-metric space being the family of bounded functions defined on complete G-metric space $(M,\hat{G})$.
Let

$$f(l) = \{1 lisrational \\ 0 liirrational$$

and

$$g(m) = \{-1 uisrational \\ 0 uisirrational.$$

Let the mapping D be defined as $Df = f^2$ for all $f \in \Im$.
Then, we only need to show that the mapping D is a D-contraction mapping.
For this, we have

$$G^*(Df, Dg, Dg) = sup\{\hat{G}(Df(l), Dg(m), Dg(m))|l, m \in M\}$$

$$= sup\{|f^2(l) - g^2(m)||l, m \in M\}$$

$$\leq sup\{|f(l) - g(m)||l, m \in M\} where 0 \leq \lambda < 1$$

$$\Rightarrow G^*(Df, Dg, Dg) \leq \lambda G^*(f, g, g).$$

Since all the conditions required for Theorem 3.3 are fulfilled, therefore, there exists a unique fixed function of D. In this example, f^2, f^4, f^6 etc. yield same fixed function of D.

Example 3.5. Let $M = [0, 1]$ and $\hat{G}$ be the G-metric defined on M. Let $\Im = C[0, 1]$ (i.e. set of all real valued continuous functions defined on [0, 1]) and the mapping $D : \Im \leftarrow \Im$ be defined as

$$Df(l) = \tfrac{3}{5} f(l) for all f \in \Im and l \in [0, 1].$$

Here, $(M, \hat{G})$ is a complete G-metric space and $\Im = C[0, 1]$ is the collection of all real valued continuous (and hence bounded) functions defined on $M = [0, 1]$.
Let $f_p(l) = \frac{l^p}{p}$ for all $p \in [0, 1]$.
Then $\{f_p(p)\}_{(p \in [0,1])}$ is a uniformly convergent sequence in $\Im$ and therefore is a cauchy sequence.
Also, the given mapping is a D-contraction mapping as

$$\begin{gathered} G^*(Df, Dg, Dg) = G^*(\tfrac{3}{5}f, \tfrac{3}{5}g, \tfrac{3}{5}g) \\ = \sup\{\hat{G}(\tfrac{3}{5}f(l), \tfrac{3}{5}g(m), \tfrac{3}{5}g(m))|l, m \in M\} \\ = \tfrac{3}{5}\sup\{\hat{G}(f(l), g(m), g(m))|l, m \in M\} \\ < \lambda G^*(f, g, g) for \tfrac{3}{5} < \lambda < 1. \end{gathered}$$

Since all the conditions required for Theorem 3.3 are fulfilled, therefore, there exists a unique fixed function of D. In this example, null function is a unique fixed function.

Theorem 3.6. *Let $(M, \hat{G})$ be a complete G-metric space (where $\hat{G}$ is the G-metric as defined earlier) and D be the collection of all bounded functions f defined on M with G-metric G^* (as defined in(3.1)).*
Also, let D be the modified D-contraction mapping on $\Im$ satisfying

$$G^*(Df, Dg, Dg) \leq \alpha G^*(f, Df, Df) + \beta G^*(g, Dg, Dg) + \gamma G^*(f, g, g)$$

for all $f, g \in \Im; \alpha, \beta, \gamma$ are non negative with $\alpha + \beta + \gamma < 1$. Then D has a unique fixed function.

Proof. Let us define a sequence $\{f_p\}_{(p \in N)}$ of functions of $\Im$ in the following way:
Let $f_0 \in \Im$ be any arbitrary function and $f_p = Df_{p-1} = D^n f_0$.

Step I: $\{f_p\}_{\{}p \in N\}$ *is a cauchy sequence in* $\Im$.

$$G^*(f_1, f_2, f_2) = G^*(Df_0, Df_1, Df_1)$$

$$\leq \alpha G^*(f_0, Df_0, Df_0) + \beta G^*(f_1, Df_1, Df_1) + \gamma G^*(f_0, f_1, f_1)$$

$$= \alpha G^*(f_0, f_1, f_1) + \beta G^*(f_1, f_2, f_2) + \gamma G^*(f_0, f_1, f_1)$$

$$= (\alpha + \gamma) G^*(f_0, f_1, f_1) + \beta G^*(f_1, f_2, f_2)$$

ISSN 2515-8260 Volume 08, Issue 03, 2021

$$\Rightarrow (1-\beta)G^*(f_1,f_2,f_2) \leq (\alpha+\gamma)G^*(f_0,f_1,f_1)$$

$$\Rightarrow G^*(f_1,f_2,f_2) \leq (\frac{\alpha+\gamma}{1-\beta})G^*(f_1,f_2,f_2)(where \beta < 1)$$

Similarly

$$G^*(f_2,f_3,f_3) \leq (\frac{\alpha+\gamma}{1-\beta})G^*(f_1,f_2,f_2)$$

$$\leq (\frac{\alpha+\gamma}{1-\beta})^2 G^*(f_0,f_1,f_1)$$

and so on.
As $(\frac{\alpha+\gamma}{1-\beta}) < 1$ and $f_0, f_1 \in \Im$ are bounded, therefore, $\{f_p\}_(p \in N)$ is a cauchy sequence in $\Im$.
Since $\Im$ is complete being the family of bounded functions defined on complete G-metric space $(M, \hat{G})$, therefore, the sequence $\{f_p\}_(p \in N)$ is convergent in $\Im$ (say it converges to $f \in \Im$).

Step II: *Existence of fixed function.*
Now it will be shown that f is a fixed function of D. Let s be any arbitrary +ve integer.

$$G^*(f,Df,Df) \leq G^*(f,f_s,f_s) + G^*(f_s,Df,Df)$$

$$= G^*(f,f_s,f_s) + G^*(Df_{s-1},Df,Df)$$

$$= G^*(f,f_s,f_s) + G^*(Df,Df_{s-1},Df_{s-1})$$

$$\Rightarrow G^*(f,Df,Df) \leq$$
$$G^*(f,f_s,f_s) + \alpha G^*(f,Df,Df) + G^*(f_{s-1},Df_{s-1},Df_{s-1}) + \gamma G^*(f,Df_{s-1},Df_{s-1})$$

$$\Rightarrow (1-\alpha)G^*(f,Df,Df) \leq G^*(f,f_s,f_s) + \beta G^*(f_{s-1},Df_{s-1},Df_{s-1}) + \gamma G^*(f,Df_{s-1},Df_{s-1})$$

The right side expression can be made arbitrary small enough by taking s sufficiently large. Thus

$$0 \leq G^*(f,Df,Df) < \epsilon$$

$$\Rightarrow G^*(f,Df,Df) = 0 i.e f is a fixed function of \Im.$$

Step III: *Uniqueness of fixed function.*
Suppose $g \in \Im$ be another fixed function of $D i.e Dg = g$ and $g \nsim f$.
Then

$$G^*(f,g,g) = G^*(Df,Dg,Dg)$$

$$\leq \alpha G^*(f,Df,Df) + \beta G^*(g,Dg,Dg) + \gamma G^*(f,g,g))$$

$$\Rightarrow (1-\gamma)G^*(f,g,g) \leq 0 (where \gamma < 1)$$

$$\Rightarrow G^*(f,g,g) \leq 0$$

which is a contraction to our assumption. This implies that f is unique. □

In this paper, we have extended the concept of $\alpha - \psi$ contractive mapping in the following manner:

Definition 3.7. The mapping $D : \Im \to \Im$ is said to be an $\alpha - \psi$ contractive mapping if there exists two functions $\alpha : M \times M \to [0, +\infty)$ and $\psi \in \Psi$ satisfying

$$\alpha(f(l), g(m), g(m))G^*(Df, Dg, Dg) \leq \psi(G^*(f, g, g) \tag{3.3}$$

for every $f, g \in \Im$ and $l, m \in M$.

Definition 3.8. Let $D : \Im \to \Im$ and $\alpha : M \times M \to [0, +\infty)$. The mapping D is called an α-admissible mapping if

$$\alpha(f(l), g(m), g(m)) \geq 1 \Rightarrow \alpha(G^*(Df, Dg, Dg)) \geq 1$$

for every $f, g \in \Im$ and $l, m \in M$.

Theorem 3.9. *Let $(M, \hat{G})$ be a complete G-metric space and $\Im$ be the collection of all bounded functions f (defined on M) with G- metric G^* (as defined in (3.1))). Let D : $\Im \to \Im$ be an $\alpha - \psi$ contractive mapping. Also, suppose that*
(i) D is α- admissible.
(ii) there is some $f_0 \in \Im$ for which $\alpha(f_0(l), Df_0(m), Df_0(m)) \geq 1$ for all $l, m \in M$.
(iii) D is continuous.
Then D possesses a fixed function in $\Im$.

Proof. Let $f_0 \in \Im$ be a function such that

$$\alpha(f_0(l), Df_0(m), Df_0(m)) \geq 1 forall l, m \in M.$$

Define the sequence $\{f_p\}_{(p \in N)}$ in $\Im$ by $f_{p+1} = Df_p$ for every $p \in N$. If $f_p = f_{p+1}$ for some $p \in N$, then f_p is a fixed function of D. Let us assume that $f_p \neq f_{p+1}$ for every $p \in N$.
As by condition *(i)*, D is α- admissible, therefore for all $l, m \in M$, we have

$$\alpha(f_0(l), f_1(m), f_1(m)) = \alpha(f_0(l), Df_0(m), Df_0(m)) \geq 1$$

$$\Rightarrow \alpha(Df_0(l), Df_1(m), Df_1(m)) = \alpha(f_1(l), f_2(m), f_2(m)) \geq 1$$

By mathematical induction, we get

$$\alpha(f_p(l), f_{p+1}(m), f_{p+1}(m)) \geq 1 forall p \in N and l, m \in M. \tag{3.4}$$

Using (3.3) and (3.4),

$$G^*(f_p, f_{p+1}, f_{p+1}) = G^*(Df_{p-1}, Df_p, Df_p)$$

$$\leq \alpha(f_{p-1}(l), f_p(m), f_p(m))G^*(Df_{p-1}, Df_p, Df_p)$$

$$\leq \psi(G^*(f_{p-1}, f_p, f_p))$$

Repetition of above process implies

$$G^*(f_p, f_{p+1}, f_{p+1}) \leq \psi^p(G^*(f_0, f_1, f_1)) forall p \in N.$$

Let $p > q \geq N$ for $N \in N$. Using triangular inequality, we have

$$G6*(f_q, f_p, f_p) \leq$$
$$G^*(f_q, f_{q+1}, f_{q+1}) + G^*(f_{q+1}, f_{q+2}, f_{q+2}) + G^*(f_{q+2}, f_{q+3}, f_{q+3}) + ... + G^*(f_{p-1}, f_p, f_p)$$
$$\leq \psi^q(G^*(f_0, f_1, f_1)) + \psi^{q+1}G^*(f_0, f_1, f_1) + ... + \psi^{p-1}(G^*(f_0, f_1, f_1))$$
$$= \Sigma_{a=q}^{p-1}\psi^a(G^*(f_0, f_1, f_1)).$$

As $\Sigma_{p=1}^{+\infty}\psi^p(l) < +\infty$ for each $l > 0$, so $\{f_p\}_{p\in N}$ is a Cauchy sequence in $\Im$ and being collection of bounded functions defined on complete G-metric space $(M, \hat{G}); (\Im, G^*)$ is itself a complete G-metric space. Therefore, there exists a function $f \in \Im$ such that

$$f_p \to f asp \to +\infty$$

As D is a continuous mapping, therefore, we have

$$Df_p \to Df asp \to +\infty \Rightarrow f_{p+1} \to Df asp \to +\infty.$$

Since limit of a convergent sequence is always unique, therefore, we have $f = Df$ i.e. f is a fixed function of D. This completes the proof. □

Theorem 3.10. *Adding condition (H) to the hypothesis of Theorem 3.9, we obtain the uniqueness of fixed function of D.*

Proof. Let us suppose that f^* and g^* be two fixed functions of D. From (H), there exists some $h^* \in \Im$ such that

$$\alpha(f^*(l), h^*(m), h^*(m)) \geq 1 and \alpha(g^*(l), h^*(m, h^*(m)) \geq 1 \tag{3.5}$$

Since D is α-admissible, by (a), we have

$$\alpha(f^*(l), D^u h^*(m), D^u h^*(m)) \geq 1 and \alpha(g^*(l), D^u h^*(m), D^u h^*(m)) \geq 1 for all u \in N. \tag{3.6}$$

Using (3.5) and $\alpha - \psi$ contractive condition

$$G^*(f, D^u h^*, D^u h^*) = G^*(Df^*, D(D^{u-1}h^*), D(D^{u-1}h^*))$$
$$\leq \alpha(f^*(l), D^{u-1}h^*(m), D^{u-1}h^*(m))G^*(Df^*, D(D^{u-1}h^*), D(D^{u-1}h^*))$$
$$\leq \psi(G^*(f^*, D^{u-1}h^*, D^{u-1}h^*))$$

which implies that

$$G^*(f, D^u h^*, D^u h^*) \leq \psi^u(G^*(f^*, h^*, h^*)) \text{ for all } u \in N.$$

Taking limit $u \to +\infty$, we get

$$D^u h^* \to f^*.$$

Similarly,

$$D^u h^* \to g^*.$$

Uniqueness of limit gives $f^* = g^*$. This proves the theorem. □

4 Application

The application is this section is based on best approximation of treatment plan for tumor patients getting intensity modulated radiation treatment therapy (IMRT).
In this technique, a proper dose deposition coefficient (DDC) matrix truncation has been used which improves the accuracy of the results. In these technique, a dose deposition coefficient (DDC) matrix is generally computed to approximate the dose distribution to each voxel in required volume of interest from every beamlet with unit intensity.
Fixed point approximation method is a very effective and efficient method to solve this problem. In this method, a proper DDC matrix truncation has been used that significantly improves accuracy of results. in 2013, Z. Tian et al.[17] given a fluence map optimization (FMO) model for dose calculation by splitting the DDC matrix in two components D_1 and D_2 on the basis of threshold value. Bortfeld [14] and Shepard et al. [4] also given some important techniques to produce algorithms for the problems come in tomotheraphy.
The matrix D_1 (major components) consist those values of DDC matrix which are higher than the threshold value while the minor component D_2 consist the remaining values. Actually, D_1 represents those doses which correspond to tumor area voxels (specifically) while D_2 represents scatter doses passing at large distances. The problem can be given as:

$$l^{k+1} = \arg\min l|D_1 l + \delta^k - T|, \tag{4.1}$$

$$\delta^{(k+1)} = D_2 l^{k+1} \tag{4.2}$$

The above model consist of an outer loop as well as an inner loop. Here $'k'$ is the iterative index of outer loop. Equation (4.1) represent the inner loop which has an iterative algorithm for value δ^k which is the dose value corresponding to D_2.
The matrix D_1 contain much reduced number of non-zero element as compared to the DDC full matrix. So, inner loop will converge more quickly than the original matrix. The outer loop given by equation (4.2) updates the value of (δ^{k+1}) using minor matrix D_2.
Here, T denotes the prescription dose for planned target volume (PTV) voxels and threshold dose for organs at risk (OAR) voxels. This mapping gives rise to a sequence $l^0, l^1, l^2, ...$ containing different dose distributions corresponding to a patient.
Following this concept, the treatment plan for more than a patient at a time, is presented through our results in a more effective way. The results proposed in this paper provide a very efficient and easy technique for estimation of suitable treatment plan.
In this given case, two tumor patients have been considered with different tumor levels. Let M denotes the set of all threshold intensity values (with a unit G_y) to be given on particular days and in particular sessions. A patient is getting the therapy two times a day. Days and sessions are denoted by D and S respectively.

$$\begin{aligned} M = \{&(1, D_1S_1, D_1S_1), (\tfrac{1}{2}, D_1S_2, D_1S_2), (1, D_2S_1, D_2S_1), (\tfrac{1}{2}, D_2S_2, D_2S_2) Patient - I \\ &(1, D_1S_1, D_1S_1), (2, D_1S_2, D_1S_2), (1, D_2S_1, D_2S_1), (2, D_2S_2, D_2S_2) Patient - II \end{aligned}$$

Note that M is complete being a closed and bounded subset of R^3. Let $\Im = \{f_1, f_2, f_3\}$ be the family of dose functions and each function represents different dose distributions (to tumor locations) of different tumor patients during IMRT.

$$f_1(l) = \{2l Patient - I, \\ l Patient - II;$$

$$f_2(m) = \{\tfrac{m}{3} Patient - I, \\ \tfrac{2m}{3} Patient - II$$

and

$$f_3(n) = \{\tfrac{n}{5} Patient - I, \\ \tfrac{2n}{5} Patient - II.$$

It is to be noted that $\Im$ is the family of bounded functions. Let $D : \Im \to \Im$ be the mapping defined as $Df = f^2 - 2f + 2 \forall f \in \Im$. It is required to prove that D is a D-contraction mapping. For $l, m, n \in M$, we have the following cases:
For Patient - I
Case I - When $l = m = n = 1$. Then
$|Df_1 - Df_2| = \frac{5}{9}$ and $|f_1 - f_2| = \frac{5}{3}$,
$|Df_2 - Df_3| = \frac{44}{225}$ and $|f_2 - f_3| = \frac{2}{15}$
$|Df_1 - Df_3| = \frac{9}{25}$ and $|f_1 - f_3| = \frac{9}{5}$.

Case II - $l = m = n = \frac{1}{2}$. Then
$|Df_1 - Df_2| = \frac{25}{36}$ and $|f_1 - f_2| = \frac{5}{6}$,
$|Df_2 - Df_3| = \frac{416}{3600}$ and $|f_2 - f_3| = \frac{2}{30}$
$|Df_1 - Df_3| = \frac{81}{100}$ and $|f_1 - f_3| = \frac{9}{10}$.

Case III - When $l = 1, m = n = \frac{1}{2}$. Then
$|Df_1 - Df_2| = \frac{11}{36}$ and $|f_1 - f_2| = \frac{11}{6}$,
$|Df_2 - Df_3| = \frac{416}{3600}$ and $|f_2 - f_3| = \frac{4}{60}$
$|Df_1 - Df_3| = \frac{19}{100}$ and $|f_1 - f_3| = \frac{19}{10}$.

Case IV - When $l = \frac{1}{2}, m = n = 1$. Then
$|Df_1 - Df_2| = \frac{4}{9}$ and $|f_1 - f_2| = \frac{2}{3}$,
$|Df_2 - Df_3| = \frac{44}{225}$ and $|f_2 - f_3| = \frac{2}{15}$
$|Df_1 - Df_3| = \frac{16}{25}$ and $|f_1 - f_3| = \frac{4}{5}$.

Case V - When $l = m = 1, n = \frac{1}{2}$. Then
$|Df_1 - Df_2| = \frac{5}{9}$ and $|f_1 - f_2| = \frac{5}{3}$,
$|Df_2 - Df_3| = \frac{329}{900}$ and $|f_2 - f_3| = \frac{7}{30}$
$|Df_1 - Df_3| = \frac{19}{100}$ and $|f_1 - f_3| = \frac{19}{10}$.

Case VI - When $l = m = \frac{1}{2}, n = 1$. Then
$|Df_1 - Df_2| = \frac{25}{36}$ and $|f_1 - f_2| = \frac{5}{6}$,
$|Df_2 - Df_3| = \frac{49}{900}$ and $|f_2 - f_3| = \frac{1}{30}$
$|Df_1 - Df_3| = \frac{16}{25}$ and $|f_1 - f_3| = \frac{4}{5}$.

For Patient - II

Case I - When $l = m = n = 1$. Then
$|Df_1 - Df_2| = \frac{1}{9}$ and $|f_1 - f_2| = \frac{1}{3}$,
$|Df_2 - Df_3| = \frac{56}{225}$ and $|f_2 - f_3| = \frac{4}{15}$
$|Df_1 - Df_3| = \frac{9}{25}$ and $|f_1 - f_3| = \frac{3}{5}$.

Case II - $l = m = n = 2$. Then
$|Df_1 - Df_2| = \frac{8}{9}$ and $|f_1 - f_2| = \frac{2}{3}$,
$|Df_2 - Df_3| = \frac{16}{225}$ and $|f_2 - f_3| = \frac{8}{15}$
$|Df_1 - Df_3| = \frac{24}{25}$ and $|f_1 - f_3| = \frac{6}{5}$.

Case III - When $l = 1, m = n = 2$. Then
$|Df_1 - Df_2| = \frac{1}{9}$ and $|f_1 - f_2| = \frac{1}{3}$,
$|Df_2 - Df_3| = \frac{16}{225}$ and $|f_2 - f_3| = \frac{8}{15}$
$|Df_1 - Df_3| = \frac{1}{25}$ and $|f_1 - f_3| = \frac{1}{5}$.

Case IV - When $l = 2, m = n = 1$. Then
$|Df_1 - Df_2| = \frac{8}{9}$ and $|f_1 - f_2| = \frac{4}{3}$,
$|Df_2 - Df_3| = \frac{56}{225}$ and $|f_2 - f_3| = \frac{4}{15}$
$|Df_1 - Df_3| = \frac{16}{25}$ and $|f_1 - f_3| = \frac{8}{5}$.

Case V - When $l = m = 1, n = 2$. Then
$|Df_1 - Df_2| = \frac{1}{9}$ and $|f_1 - f_2| = \frac{1}{3}$,
$|Df_2 - Df_3| = \frac{16}{225}$ and $|f_2 - f_3| = \frac{2}{15}$
$|Df_1 - Df_3| = \frac{1}{25}$ and $|f_1 - f_3| = \frac{1}{5}$.

Case VI - When $l = m = 2, n = 1$. Then
$|Df_1 - Df_2| = \frac{8}{9}$ and $|f_1 - f_2| = \frac{2}{3}$,
$|Df_2 - Df_3| = \frac{56}{225}$ and $|f_2 - f_3| = \frac{14}{15}$
$|Df_1 - Df_3| = \frac{16}{25}$ and $|f_1 - f_3| = \frac{8}{5}$.
From all above cases, for Patient-I

$$G^*(Df1, Df2, Df3) = \sup\{|Df1 - Df2| + |Df2 - Df3| + |Df1 - Df3| l, m, n \in M\}$$

$$= \tfrac{25}{36} \leq \tfrac{2}{3}\{\tfrac{11}{6} + \tfrac{7}{30} + \tfrac{19}{10}\}$$

$$= \lambda G^*(f1, f2, f3).$$

and for Patient-II

$$G^*(Df1, Df2, Df3) = \sup\{|Df1 - Df2| + |Df2 - Df3| + |Df1 - Df3| l, m, n \in M\}$$

$$= \tfrac{8}{9} \leq \tfrac{2}{3}\{\tfrac{4}{3} + \tfrac{14}{15} + \tfrac{8}{5}\}$$

$$= \lambda G^*(f1, f2, f3).$$

where $\lambda = \frac{2}{3} < 1$.
Thus, all the conditions required for Theorem 3.3 are fulfilled. Therefore, there exists a unique fixed function $f1$ of D that yields suitable doses for two patients at the same time.

5 References

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