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Few Properties of $m_{2,k}(\sigma)$ Represented By A Dirichlet Series

Praneeta Verma*

Abstract

In this paper, we have discussed some properties of the mean values of an entire function represented by Dirichlet series in the usual notation. It is obvious that generally $\lambda_* \leq \lambda$ and $\rho_* \leq \rho$, there are entire Dirichlet series for which $\lambda_* < \lambda$ and $\rho_* < \rho$. Hence, we have generally to distinguish between the limits as well as types of $f(s)$ belonging to the same order ρ_* $(0 < \rho_* < \infty)$. In this paper, we obtain some result of $m_{2,k}(\sigma)$ for the mean value of an entire Dirichlet series.

THEOREM *If* $f(s) = \sum_1^\infty a_n e^{s\lambda_n}$ *is an entire function of Ritt order* σ *and lower order* λ *then*

$$\begin{matrix}\rho_* \\ \lambda_*\end{matrix} \leq \lim_{\sigma\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log\log m_{2,k}(\sigma)}{\sigma} \leq \begin{matrix}\rho \\ \lambda\end{matrix} \quad (a)$$

Under the additional condition on $\{\lambda_n\}$,

$$0 \leq \lim_{n\to\infty} \sup \frac{\log n}{\lambda_n} = D < \infty, \quad (A)$$

(a) Becomes

$$\lim_{\sigma\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log\log m_{2,k}(\sigma)}{\sigma} = \begin{matrix}\rho \\ \lambda\end{matrix} = \begin{matrix}\rho_* \\ \lambda_*\end{matrix} \quad (b)$$

In fact, for the truth of 'lim sup' part of (b) the following condition on $\{\lambda_n\}$ *is sufficient.*

$$\lim_{n\to\infty} \frac{\log n}{\lambda_n \log \lambda_n} = 0. \quad (A')$$

Keywords: Generalized order ρ, Generalized lower order λ, Type of the function T

INTRODUCTION

In the usual notation,

$$f(s) = \sum_1^\infty a_n e^{s\lambda_n}, (s = \sigma + it),$$

$$0 < \lambda_n < \lambda_{n+1} \quad (n \geq 1) \lim_{n\to\infty} \lambda_n = \infty, \quad (1.1)$$

Is an entire function in the sense that the Dirichlet series representing it, is absolutely convergent for all finite s and possesses two generally different pairs of orders as Ritt [1] defined.

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$$\lim_{\sigma\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log\log M(\sigma)}{\sigma} = \begin{matrix}\rho \\ \lambda\end{matrix}; \tag{1.2}$$

As Sugimura ([2]) define

$$\lim_{\sigma\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log\log \mu(\sigma)}{\sigma} = \begin{matrix}\rho_* \\ \lambda_*\end{matrix}; \tag{1.3}$$

Clearly $\rho_* \le \rho$ and $\lambda_* \le \lambda$. there are entire Dirichlet series for which $\rho_* < \rho, \lambda_* < \lambda$ $([2], Satz\, 4)$.

Where $0 \le \lambda, \rho \le \infty, 0 \le \lambda_*, \rho_* \le \infty, and\ M(\sigma), \mu(\sigma)$ their usual meanings, viz.

$$M(\sigma) = \underset{-\infty<t<\infty}{l.u.b.} \left| f(\sigma+it) \right|, \quad \mu(\sigma) = \max_{n\ge 1} \left| a_n e^{(\sigma+it)\lambda_n} \right|$$

The type T, t associated with ρ and type T*, t_* associate with ρ_* are defined in the usual way as follow:

$$\lim_{\sigma\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log M(\sigma)}{e^{\rho\sigma}} = \begin{matrix}T \\ t\end{matrix}, \quad (0<\rho<\infty) \tag{1.4}$$

$$\lim_{\sigma\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log \mu(\sigma)}{e^{\rho_*\sigma}} = \begin{matrix}T_* \\ t_*\end{matrix}, \quad (0<\rho_*<\infty). \tag{1.5}$$

The mean values of $f(s)$ are defined as follows as shown in ([3] p.270)

$$\{I_2(\sigma, f)\}^2 = \{I_2(\sigma)\}^2 = A_2(\sigma) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left| f(\sigma+it) \right|^2 dt, \quad 0<\delta<\infty, \tag{1.6}$$

$$m_{2,k}(\sigma, f) = m_{2,k}(\sigma) = \lim_{T\to\infty} \frac{1}{2Te^{k\sigma}} \int_{-\infty}^{\sigma} \int_{-T}^{T} \left| f(x+it) \right|^2 e^{kx} dx\, dt$$

$$= \frac{1}{e^{k\sigma}} \int_{-\infty}^{\sigma} A_2(x) e^{kx} dx, \quad 0<k<\infty \tag{1.7}$$

Kamthan [4] has obtained a few properties of the mean $V_k(\sigma, f)$ of $f(s)$ where $V_k(\sigma, f)$ is defined as

$$V_{2,k}(\sigma, f) = \frac{1}{e^{k\sigma}} \int_{0}^{\sigma} A_2(x) e^{kx} dx = m_{2,k}(\sigma) - J, \quad 0<k<\infty \tag{1.8}$$

Where J is a real constant depending on k and f. It easily follows from (1.8) that for all large σ the behavior of $m_{2,k}(\sigma, f)$ is the same as that of $V_k(\sigma, f)$ and all results that have been derived for $V_k(\sigma, f)$ can be obtained for $m_{2,k}(\sigma, f)$.

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THEOREM

If $f(s) = \sum_{1}^{\infty} a_n e^{s\lambda_n}$ is an entire function of Ritt order σ and A lower order λ then

$$\frac{\rho_*}{\lambda_*} \leq \lim_{\sigma\to\infty} \begin{smallmatrix}\sup\\ \inf\end{smallmatrix} \frac{\log\log m_{2,k}(\sigma)}{\sigma} \leq \frac{\rho}{\lambda} \tag{2.1}$$

Under the additional condition on $\{\lambda_n\}$,

$$0 \leq \lim_{n\to\infty} \sup \frac{\log n}{\lambda_n} = D < \infty, \tag{2.2}$$

(2.1) Becomes

$$\lim_{\sigma\to\infty} \begin{smallmatrix}\sup\\ \inf\end{smallmatrix} \frac{\log\log m_{2,k}(\sigma)}{\sigma} = \frac{\rho}{\lambda} = \frac{\rho_*}{\lambda_*} \tag{2.3}$$

In fact, for the truth of 'lim sup' part of (b) the following condition on $\{\lambda_n\}$ is sufficient.

$$\lim_{n\to\infty} \frac{\log n}{\lambda_n \log \lambda_n} = 0. \tag{2.4}$$

Proof. The definition of $A_2(\sigma)$ and Parseval's identity for Dirichlet series,

$$A_2(\sigma) = \sum_{1}^{\infty} |a_n|^2 e^{2\sigma}\lambda_n,$$

Together gives us

$$\{u(\sigma)\}^2 \leq A_2(\sigma) \leq \{M(\sigma)\}^2 \tag{2.5}$$

Also, since $M(\sigma)$ is increasing function of σ,

$$m_{2,k}(\sigma) = \frac{1}{e^{k\sigma}} \int_{-\infty}^{\sigma} A_2(x) e^{kx} dx, \leq \frac{\{M(\sigma)\}^2}{k} \tag{2.6}$$

This leads to

$$\lim_{\sigma\to\infty} \begin{smallmatrix}\sup\\ \inf\end{smallmatrix} \frac{\log\log m_{2,k}(\sigma)}{\sigma} \leq \lim_{\sigma\to\infty} \begin{smallmatrix}\sup\\ \inf\end{smallmatrix} \frac{\log\log M(\sigma)}{\sigma} \tag{2.7}$$

Comparing (1.2) and (2.7), we get

$$\lim_{\sigma\to\infty} \begin{smallmatrix}\sup\\ \inf\end{smallmatrix} \frac{\log\log m_{2,k}(\sigma)}{\sigma} \leq \frac{\rho}{\lambda} \tag{2.8}$$

From (1.7), we have for h>0

$$m_{2,k}(\sigma+h) \geq \frac{1}{e^{k(\sigma+h)}} \int_{\sigma}^{\sigma+h} A_2(x) e^{kx} dx,$$

This with (2.5) will give us

$$m_{2,k}(\sigma+h) \geq \frac{\{\mu(\sigma)\}^2}{k}\{1-e^{-kh}\} \qquad (2.9)$$

Consequently, we get

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log \mu(\sigma)}{\sigma} \leq \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log m_{2,k}(\sigma)}{\sigma}$$

Now using (1.3), we get

$$\begin{matrix}\rho_*\\ \lambda_*\end{matrix} \leq \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log m_{2,k}(\sigma)}{\sigma} \qquad (2.10)$$

Combining (2.8) and (2.10), we have

$$\begin{matrix}\rho_*\\ \lambda_*\end{matrix} \leq \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log m_{2,k}(\sigma)}{\sigma} \leq \begin{matrix}\rho\\ \lambda\end{matrix}$$

To prove (2.3) we use the known results ([5], p.68) that under the condition (2.2)

$$M(\sigma) < K\mu(\sigma + D + \epsilon)(2.11) \qquad (2.11)$$

Where ε is an arbitrary small positive number r and k is constant depending on D, ε (2.6) in conjunction with (2.11) gives

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log m_{2,k}(\sigma)}{\sigma} \leq \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(\sigma)}{\sigma} \leq \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log \mu(\sigma)}{\sigma}$$

From this particular case, stated as part of the theorem now follows immediately
It is known that under condition

$$\lim_{n\to\infty} \frac{\log n}{\lambda_n \log \lambda_n} = 0$$

$$\rho = \lim_{n\to\infty} \sup \frac{\lambda_n \log \lambda}{\log |a_n|^{-1}} \quad [6]$$

Further, from the result of Reddy [7] we conclude that

$$\rho_* = \lim_{n\to\infty} \sup \frac{\lambda_n \log \lambda}{\log |a_n|^{-1}}$$

Combining these two, we have

$$\rho_* = \rho$$

Thus, we have completed the proof of the theorem.

CONCLUSION

Juneja [8] has proved the particular case of our theorem under the condition (2.2) with D = 0. He has used the asymptotic equality $log M(\sigma) \sim log \mu(\sigma), \sigma \to \infty$. The method of proofs of our results is different from that of Juneja. For a sufficient condition for the truth of asymptotic equality is known only in the form ([9], p.73)

$$0 \leq \lim_{n\to\infty} \sup \frac{\log n}{\log \lambda_n} = E < \infty$$

In addition, this condition is not necessarily implied by Juneja's assumption of (2.2) with D = 0.

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